35-th Spanish Mathematical Olympiad 1999

Second Round

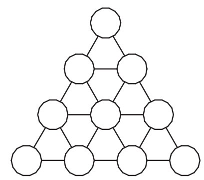
Granada

First Part

1. The lines t and t', tangent to the parabola $y = x^2$ at points A and B respectively, intersect at point C. The median of triangle ABC from C has length m. Find the area of $\triangle ABC$ in terms of m.

2. Prove that there exists a sequence of positive integers $a_1, a_2, a_3, \dots$ such that $a_1^2 + a_2^2 + \cdots + a_n^2$ is a perfect square for all positive integers n.

3. A one player game is played on the triangular board shown on the picture. A token is placed on each circle. Each token is white on one side and black on the other. Initially, the token at one vertex of the triangle has the black side up, while the others have the white sides up. A move consists of removing a token with the black side up and turning over the adjacent tokens (two tokens are adjacent if they are joined by a segment). Is it possible to remove all the tokens by a sequence of moves?



Second Part

4. A box contains 900 cards, labeled from 100 to 999. Cards are removed one at a time without replacement. What is the smallest number of cards that must be removed to guarantee that the labels of at least three removed cards have equal sums of digits?

5. The distance from the baricenter G of a triangle ABC to its sides a, b, c are denoted g_a, g_b, g_c respectively. Let r be the inradius of the triangle. Prove that:

   (a) $g_a, g_b, g_c \geq \frac{2}{3} r$;

   (b) $g_a + g_b + g_c \geq 3r$.

6. A plane is divided into n regions by three families of parallel lines. No three lines pass through the same point. What is the smallest number of lines needed so that $N > 1999$?

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com