32-nd Spanish Mathematical Olympiad 1996

Second Round

Tarragona

First Part

1. The natural numbers a and b are such that $\frac{a+1}{b}+\frac{b+1}{a}$ is an integer. Show that the greatest common divisor of a and b is not greater than $\sqrt{a+b}$.

2. Let G be the centroid of a triangle ABC. Prove that if $AB+GC=AC+GB$, then the triangle is isosceles.

3. Consider the functions $f(x)=ax^2+bx+c$, $g(x)=cx^2+bx+a$, where a,b,c are real numbers. Given that $|f(-1)|\le 1$, $|f(0)|\le 1$, $|f(1)|\le 1$, prove that
$$|f(x)|\le \frac{5}{4} \text{ and } |g(x)|\le 2 \quad \text{for } -1\le x\le 1.$$

Second Part

4. For each real value of p, find all real solutions of the equation
$$\sqrt{x^2-p}+2\sqrt{x^2-1}=x.$$

5. At Port Aventura there are 16 secret agents, each of whom is watching one or more other agents. It is known that if agent A is watching agent B, then B is not watching A. Moreover, any 10 agents can be ordered so that the first is watching the second, the second is watching the third, etc, the last is watching the first. Show that any 11 agents can also be so ordered.

6. A regular pentagon is constructed externally on each side of a regular pentagon of side 1. The figure is then folded and the two edges of the external pentagons meeting at each vertex of the original pentagon are glued together. Find the volume of water that can be poured into the obtained container.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com