30-th Spanish Mathematical Olympiad 1994

Second Round

Madrid

First Part

1. Prove that if an arithmetic progression contains a perfect square, then it contains infinitely many perfect squares.

2. Let $Oxyz$ be a trihedron whose edges x,y,z are mutually perpendicular. Let C be the point on the ray z with $OC = c$. Points P and Q vary on the rays x and y respectively in such a way that $OP + OQ = k$ is constant. For every P and Q, the circumcenter of the sphere through O,C,P,Q is denoted by W. Find the locus of the projection of W on the plane Oxy. Also find the locus of points W.

3. A tourist office was investigating the numbers of sunny and rainy days in a year in each of six regions. The results are partly shown in the following table:

Region	sunny or rainy	unclassified
A	336	29
B	321	44
C	335	30
D	343	22
E	329	36
F	330	35

Looking at the detailed data, an officer observed that if one region is excluded, then the total number of rainy days in the other regions equals one third of the total number of sunny days in these regions. Determine which region is excluded.

Second Part

4. In a triangle ABC with $\angle A = 36^\circ$ and $AB = AC$, the bisector of the angle at C meets the oposite side at D. Compute the angles of $\triangle BCD$. Express the length of side BC in terms of the length b of side AC without using trigonometric functions.

5. Let 21 pieces, some white and some black, be placed on the squares of a 3×7 rectangle. Prove that there always exist four pieces of the same color standing at the vertices of a rectangle.

6. A convex n-gon is dissected into m triangles such that each side of each triangle is either a side of another triangle or a side of the polygon. Prove that $m+n$ is even. Find the number of sides of the triangles inside the square and the number of vertices inside the square in terms of m and n.