27-th Spanish Mathematical Olympiad 1991

Second Round

First Part

1. In the coordinate plane, consider the set of all segments of integer lengths whose endpoints have integer coordinates. Prove that no two of these segments form an angle of 45°. Are there such segments in coordinate space?

2. Given two distinct elements $a, b \in \{-1, 0, 1\}$, consider the matrix

$$A = \begin{bmatrix} a+b & a+b^2 & \cdots & a+b^m \\ a^2+b & a^2+b^2 & \cdots & a^2+b^m \\ \cdots & \cdots & \cdots & \cdots \\ a^n+b & a^n+b^2 & \cdots & a^n+b^m \end{bmatrix}.$$

   Find a subset S of the set of the rows of A, of minimum size, such that every other row of A is a linear combination of the rows in S with integer coefficients.

3. What condition must be satisfied by the coefficients u, v, w if the roots of the polynomial $x^3 - ux^2 + vx - w$ are the sides of a triangle?

Second Part

4. The incircle of ABC touches the sides BC, CA, AB at A', B', C' respectively. The line $A'C'$ meets the angle bisector of $\angle A$ at D. Find $\angle ADC$.

5. For a positive integer n, let $s(n)$ denote the sum of the binary digits of n. Find the sum $s(1) + s(2) + s(3) + \cdots + s(2^k)$ for each positive integer k.

6. Find the integer part of

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{1000}}.$$

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