23-rd Spanish Mathematical Olympiad 1987

Second Round

February 1987

First Part

1. Let a, b, c be the side lengths of a scalene triangle and let O_a, O_b and O_c be three concentric circles with radii a, b and c respectively.
   (a) How many equilateral triangles with different areas can be constructed such that the lines containing the sides are tangent to the circles?
   (b) Find the possible areas of such triangles.

2. Show that for each natural number $n > 1$
$$1\sqrt{\binom{n}{1}}+2\sqrt{\binom{n}{2}}+\cdots+n\sqrt{\binom{n}{n}}<\sqrt{2^{n-1}n^3}$$

3. A given triangle is divided into n triangles in such a way that any line segment which is a side of a tiling triangle is either a side of another tiling triangle or a side of the given triangle. Let s be the total number of sides and v be the total number of vertices of the tiling triangles (counted without multiplicity).
   (a) Show that if n is odd then such divisions are possible, but each of them has the same number v of vertices and the same number s of sides. Express v and s as functions of n.
   (b) Show that, for n even, no such tiling is possible.

Second Part

4. If a and b are distinct real numbers, solve the systems
$$\text{(a)}\ \begin{cases} x+y=1 \\ (ax+by)^2 \le a^2x+b^2y \end{cases} \quad \text{and} \quad \text{(b)}\ \begin{cases} x+y=1 \\ (ax+by)^4 \le a^4x+b^4y. \end{cases}$$

5. In a triangle ABC, D lies on AB, E lies on AC and $\angle ABE = 30^\circ$, $\angle EBC = 50^\circ$, $\angle ACD = 20^\circ$, $\angle DCB = 60^\circ$. Find $\angle EDC$.

6. For all natural numbers n, consider the polynomial $P_n(x) = x^{n+2} - 2x + 1$.
   (a) Show that the equation $P_n(x) = 0$ has exactly one root c_n in the open interval $(0,1)$.
   (b) Find $\lim\limits_{n\to\infty} c_n$.

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