22-nd Spanish Mathematical Olympiad 1986

Second Round

Madrid, February 1986

First Part

1. Define the distance between real numbers x and y by

$$d(x,y) = \sqrt{([x]-[y])^2 + (\{x\}-\{y\})^2}.$$

Determine (as a union of intervals) the set of real numbers whose distance from $\frac{3}{2}$ is less than $\frac{202}{100}$.

2. A segment d is said to divide a segment s if there is a natural number n such that $s = nd = d + d + \cdots + d$ (n times).

   (a) Prove that if a segment d divides segments s and s' with $s < s'$, then it also divides their difference $s' - s$.

   (b) Prove that no segment divides the side s and the diagonal s' of a regular pentagon (consider the pentagon formed by the diagonals of the given pentagon without explicitly computing the ratios).

3. Find the natural numbers n for which $5^n + 3$ is a power of 2.

Second Part

4. Denote by $m(a,b)$ the arithmetic mean of positive real numbers a,b. Given a positive real function g having positive derivatives of the first and second order, define the mean value of a and b with respect to g by

$$2g(\mu(a,b)) = g(a) + g(b).$$

Decide which of the two mean values m and μ is larger.

5. Consider the curve Γ defined by the equation $y^2 = x^3 + bx + b^2$, where b is a nonzero rational constant. Inscribe in the curve Γ a triangle whose vertices have rational coordinates.

6. Evaluate $\displaystyle\prod_{k=1}^{14} \cos\frac{k\pi}{15}$.

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