20-th Spanish Mathematical Olympiad 1984

Second Round

Madrid, February 1984

First Part

1. At a position O of an airport in a plateau there is a gun which can rotate arbitrarily. Two tanks moving along two given segments AB and CD attack the airport. Determine, by a ruler and a compass, the reach of the gun, knowing that the total length of the parts of the trajectories of the two tanks reachable by the gun is equal to a given length l.

2. Find the number of five-digit numbers whose square ends in the same five digits in the same order.

3. If p and q are positive numbers with $p+q=1$, knowing that any real numbers x,y satisfy $(x-y)^2 \geq 0$, show that
$$\frac{x+y}{2} \geq \sqrt{xy}, \quad \frac{x^2+y^2}{2} \geq \left(\frac{x+y}{2}\right)^2,$$
$$\left(p+\frac{1}{p}\right)^2 + \left(q+\frac{1}{q}\right)^2 \geq \frac{25}{2}.$$

4. Evaluate $\lim_{n\to\infty} \cos\frac{x}{2}\cos\frac{x}{2^2}\cos\frac{x}{2^3}\cdots\cos\frac{x}{2^n}$.

Second Part

5. Let A and A' be fixed points on two equal circles in the plane and let AB and $A'B'$ be arcs of these circles of the same length x. Find the locus of the midpoint of segment BB' when x varies:
   (a) if the arcs have the same direction;
   (b) if the arcs have opposite directions.

6. Consider the circle γ with center at point $(0,3)$ and radius 3, and a line r parallel to the axis Ox at a distance 3 from the origin. A variable line through the origin meets γ at point M and r at point P. Find the locus of the intersection point of the lines through M and P parallel to Ox and Oy respectively.

7. Consider the natural numbers written in the decimal system.
   (a) Find the smallest number which decreases five times when its first digit is erased. Which form do all numbers with this property have?
   (b) Prove that there is no number that decreases 12 times when its first digit is erased.

(c) Find the necessary and sufficient condition on k for the existence of a natural number which is divided by k when its first digit is erased.

8. Find the remainder upon division by $x^2 - 1$ of the determinant

$$\begin{vmatrix} x^3+3x & 2 & 1 & 0 \\ x^2+5x & 3 & 0 & 2 \\ x^4+x^2+1 & 2 & 1 & 3 \\ x^5+1 & 1 & 2 & 3 \end{vmatrix}.$$

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