Slovenian National Mathematical Olympiad 1999

Final Round

Celje, May 15–16, 1999

1-st Grade

1. Two three-digit numbers are given. The hundreds digit of each of them is equal to the units digit of the other. Find these numbers if their difference is 297 and the sum of digits of the smaller number is 23.

2. Find all integers x, y such that $2x + 3y = 185$ and $xy > x + y$.

3. The incircle of a right triangle ABC touches the hypotenuse AB at a point D. Show that the area of $\triangle ABC$ equals $AD \cdot DB$.

4. On a mountain, three shepherds cyclically alternate shearing the same herd of sheeps. The shepherds agreed to obey the following rules:

   (i) Every day a sheep can be shorn on one side only;

   (ii) Every day at least one sheep must be shorn;

   (iii) No two days the same group of sheeps can be shorn.

   The shepherd who first breaks the agreement will have to accompany the herd in the valley next fall. Can anyone of the shepherds shear the sheeps in such a way to make sure that he will avoid this punishment?

2-nd Grade

1. Prove that the product of three consecutive positive integers is never a perfect square.

2. Three unit vectors a, b, c are given on the plane. Prove that one can choose the signs in the expression $x = \pm a \pm b \pm c$ so as to obtain a vector x with $|x| \leq \sqrt{2}$.

3. A semicircle with diameter AB is given. Two non-intersecting circles k_1 and k_2 with different radii touch ther diameter AB and touch the semicircle internally at C and D, respectively. An interior common tangent t of k_1 and k_2 touches k_1 at E and k_2 at F. Prove that the lines CE and DF intersect on the semicircle.

4. Three integers are written on a blackboard. At every step one of them is erased and the sum of the other two decreased by 1 is written instead. Is it possible to obtain the numbers $17, 75, 91$ if the three initial numbers were: (a) $2, 2, 2$; (b) $3, 3, 3$?

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3-rd Grade

1. What is the smallest possible value of $|12^m - 5^n|$, where m and n are positive integers?

2. Consider the polynomial $p(x) = x^{1999} + 2x^{1998} + 3x^{1997} + \cdots + 2000$. Find a nonzero polynomial whose roots are the reciprocal values of the roots of $p(x)$.

3. Let O be the circumcenter of a triangle ABC, P be the midpoint of AO, and Q be the midpoint of BC. If $\angle ABC = 4\angle OPQ$ and $\angle ACB = 6\angle OPQ$, compute $\angle OPQ$.

4. A pawn is put on each of $2n$ arbitrary selected cells of an $n \times n$ board ($n > 1$). Prove that there are four cells that are marked with pawns and whose centers form a parallelogram.

4-th Grade

1. Let $r_1, r_2, \dots, r_m$ be positive rational numbers with the sum 1. Find the minimum and maximum values of the function $f : \mathbb{N} \to \mathbb{Z}$ defined by

$$f(n) = n - [r_1 n] - [r_2 n] - \cdots - [r_m n].$$

2. The numbers $1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{1999}$ are written on a blackboard. In every step we choose two of them, say a and b, erase them and write the number $ab + a + b$ instead. This step is repeated until only one number remains. Can the last remaining number be equal to 2000?

3. A section of a rectangular parallelepiped by a plane is a regular hexagon. Prove that this parallelepiped is a cube.

4. Let be given three-element subsets $A_1, A_2, \dots, A_6$ of a six-element set X. Prove that the elements of X can be colored with two colors in such a way that none of the given subsets is monochromatic.

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