Slovenian National Mathematical Olympiad 2005

Final Round

Velenje, April 16, 2005

1-st Grade

1. If x,y,z are real numbers such that $xyz = 1$, evaluate

$$\frac{x+1}{xy+x+1} + \frac{y+1}{yz+y+1} + \frac{z+1}{zx+z+1}.$$

2. Find all prime numbers p for which the number p^2+11 has less than 11 divisors.

3. Suppose that a triangle ABC with incenter I satisfies $CA + AI = BC$. Find the ratio between the measures of the angles $\angle BAC$ and $\angle CBA$.

4. The friends Alex, Ben and Charles prepared a lot of labels and wrote one of the numbers $2,3,4,5,6,7,8$ on each label. Then Mary joined them and glued one label onto the forehead of each friend. Of course, each of the friends can see the labels on the others' foreheads, but not the one on his own forehead. Mary told them: ”The numbers on your foreheads are not all distinct, and their product is a perfect square.” Can any of the friends find out the number on his forehead?

2-nd Grade

1. Find all real numbers x,y such that $x^3 - y^3 = 7(x-y)$ and $x^3 + y^3 = 5(x+y)$.

2. For which prime numbers p and q is $(p+1)^q$ a perfect square?

3. Let T be a point inside a square $ABCD$. The lines TA, TB, TC, TD meet the circumcircle of $ABCD$ again at A', B', C', D', respectively. Prove that $A'B' \cdot C'D' = A'D' \cdot B'C'$.

4. The village chatterboxes are exchanging their gossip by phone every day so that any two of them talk to each other exactly once. A certain day, every chatterbox called up at least one of the other chatterboxes. Show that there exist three chatterboxes such that the first called up the second, the second called up the third, and the third called up the first.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com

3-rd Grade

1. Evaluate the sum $[\log_2 1]+[\log_2 2]+[\log_2 3]+\cdots+[\log_2 256]$.

2. Find the smallest prime number p for which the number p^3+2p^2+p has exactly 42 divisors.

3. In an isosceles triangle ABC with $AB=AC$, D is the midpoint of AC and E is the projection of D onto BC. Let F be the midpoint of DE. Prove that the lines BF and AE are perpendicular if and only if the triangle ABC is equilateral.

4. Several teams from Littletown and Bigtown took part on a tournament. There were nine more teams from Bigtown than those from Littletown. Any two teams played exactly one match, and the winner and loser got 1 and 0 points respectively (no ties). The teams from Bigtown in total gained nine times more points than those from Littletown. What is the maximum possible number of wins of the best team from Littletown?

4-th Grade

1. Find all positive numbers x such that $20\{x\}+0.5[x]=2005$.

2. Let (a_n) be a geometrical progression with positive terms. Define $S_n=\log a_1+\log a_2+\cdots+\log a_n$. Prove that if $S_n=S_m$ for some $m\neq n$, then $S_{n+m}=0$.

3. The tangent lines from a point P meet a circle k at A and B. Let X be an arbitrary point on the shorter arc AB, and C and D be the orthogonal projections of P onto the lines AX and BX, respectively. Prove that the line CD passes through a fixed point Y as X moves along the arc AB.

4. William was bored at the math lesson, so he drew a circle and $n\geq 3$ empty cells around the circumference. In every cell he wrote a positive number. Later on he erased the numbers and in every cell wrote the geometric mean of the numbers previously written in the two neighboring cells. Show that there exists a cell whose number was not replaced by a larger number.