# Slovenian National Mathematical Olympiad 2001

## Final Round

Idrija, May 12–13, 2001

### 1-st Grade

1. None of positive integers $k,m,n$ is divisible by 5. Prove that at least one of the numbers $k^2-m^2$, $m^2-n^2$, $n^2-k^2$ is divisible by 5.

2. Tina wrote a positive number on each of five pieces of paper. She did not say which numbers she wrote, but revealed their pairwise sums instead: $17,20,28,14,42,36,28,39,25,31$. Which numbers did she write?

3. For an arbitrary point $P$ on a given segment $AB$, two isosceles right triangles $APQ$ and $PBR$ with the right angles at $Q$ and $R$ are constructed on the same side of the line $AB$. Prove that the distance from the midpoint $M$ of $QR$ to the line $AB$ does not depend on the choice of $P$.

4. Andrej and Barbara play the following game with two strips of newspaper of length $a$ and $b$. They alternately cut from any end of any of the strips a piece of length $d$. The player who cannot cut such a piece loses the game. Andrej allows Barbara to start the game. Find out how the lengths of the strips determine the winner.

### 2-nd Grade

1. Determine all positive integers $a,b,c$ such that $ab+ac+bc$ is a prime number and
$$\frac{a+b}{a+c}=\frac{b+c}{b+a}.$$

2. Let $p(n)$ denote the product of decimal digits of a positive integer $n$. Compute the sum $p(1)+p(2)+\cdots+p(2001)$.

3. Let $E$ and $F$ be points on the side $AB$ of a rectangle $ABCD$ such that $AE=EF$. The line through $E$ perpendicular to $AB$ intersects the diagonal $AC$ at $G$, and the segments $FD$ and $BG$ intersect at $H$. Prove that the areas of the triangles $FBH$ and $GHD$ are equal.

4. Find the smallest number of squares on an $8\times 8$ board that should be colored so that every $L$-tromino on the board contains at least one colored square.



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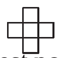
## 3-rd Grade

1. (a) Prove that $\sqrt{n+1}-\sqrt{n}<\frac{1}{2\sqrt{n}}<\sqrt{n}-\sqrt{n-1}$ for all $n \in \mathbb{N}$.

   (b) Prove that the integer part of the sum $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{m^2}}$, where $m \in \mathbb{N}$, is either $2m-2$ or $2m-1$.

2. Find all rational numbers $r$ such that the equation $rx^2+(r+1)x+r=1$ has integer solutions.

3. A point $D$ is taken on the side $BC$ of an acute-angled triangle $ABC$ such that $AB=AD$. Point $E$ on the altitude from $C$ of the triangle is such that the circle $k_1$ with center $E$ is tangent to the line $AD$ at $D$. Let $k_2$ be the circle through $C$ that is tangent to $AB$ at $B$. Prove that $A$ lies on the line determined by the common chord of $k_1$ and $k_2$.

4. Cross-shaped tiles  are to be placed on a $8\times 8$ square grid without overlapping. Find the largest possible number of tiles that can be placed.

## 4-th Grade

1. Let $a,b,c,d,e,f$ be positive numbers such that $a,b,c,d$ is an arithmetic progression, and $a,e,f,d$ is a geometric progression. Prove that $bc \geq ef$.

2. Find all prime numbers $p$ for which $3^p-(p+2)^2$ is also prime.

3. Let $D$ be the foot of the altitude from $A$ in a triangle $ABC$. The angle bisector at $C$ intersects $AB$ at a point $E$. Given that $\angle CEA=\pi/4$, compute $\angle EDB$.

4. Let $n\geq 4$ points on a circle be denoted by 1 through $n$. A pair of two non-adjacent points denoted by $a$ and $b$ is called *regular* if all numbers on one of the arcs determined by $a$ and $b$ are less than $a$ and $b$. Prove that there are exactly $n-3$ regular pairs.