Slovenian National Mathematical Olympiad 2000

Final Round

Novo Mesto, May 13–14, 2000

1-st Grade

1. In the expression $4 \cdot RAKEC = CEKAR$, each letter represents a (decimal) digit. Replace the letters so that the equality is true.

2. Find all real numbers a for which the following equation has a unique real solution:
$$|x-1|+|x-2|+\cdots+|x-99|=a.$$

3. Let ABC be a triangle such that the altitude CD is equal to AB. The squares $DBEF$ and $ADGH$ are constructed with F,G on CD. Show that the segments CD,AE and BH are concurrent.

4. All vertices of a convex n-gon ($n \geq 3$) in the plane have integer coordinates. Show that its area is at least $\dfrac{n-2}{2}$.

2-nd Grade

1. Let n be the number of ordered 5-tuples $(a_1,a_2,\dots,a_5)$ of positive integers such that $\dfrac{1}{a_1}+\dfrac{1}{a_2}+\cdots+\dfrac{1}{a_5}=1$. Is n an even number?

2. Three students start walking with constant speeds at the same time, each along a straight line in the plane. Prove that if the students are not on the same line at the beginning, then they will be on the same line at most twice during their journey.

3. A point D is taken inside an isoscles triangle ABC with base AB and $\angle C = 80^\circ$ such that $\angle DAB = 10^\circ$ and $\angle DBA = 20^\circ$. Compute $\angle ACD$.

4. Alex and Jack have 1000 sheets each. Each of them writes the numbers $1,\dots,2000$ on his sheets in an arbitrary order, with one number on each side of a sheet. The sheets are to be placed on the floor so that one side of each sheet is visible. Prove that they can do so in such a way that each of the numbers from 1 to 2000 is visible.

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3-rd Grade

1. Find all prime numbers whose base b representations (for some b) contain each of the digits $0,1,\dots,b-1$ exactly once. (Digit 0 may appear as the first digit.)

2. Consider the polynomial $p(x) = a_n x^n + \dots + a_1 x + a_0$ with real coefficients such that $0 \le a_i \le a_0$ for each $i = 1,2,\dots,n$. If a is the coefficient at x^{n+1} in the polynomial $q(x) = p(x)^2$, prove that $2a \le p(1)^2$.

3. Let H be the orthocenter of an acute-angled triangle ABC with $AC \neq BC$. The line through the midpoints of the segments AB and HC intersects the bisector of $\angle ACB$ at D. Suppose that the line HD contains the circumcenter of $\triangle ABC$. Determine $\angle ACB$.

4. A pile of 2000 coins is given on a table. In each step we choose a pile with at least three coins, remove one coin from it, and divide the rest of this pile into two piles (not necessarily of the same size). Is it possible that after several steps each pile on the table has exactly three coins?

4-th Grade

1. The sequence (a_n) is given by $a_1 = 2$, $a_2 = 500$, $a_3 = 2000$ and
$$\frac{a_{n+2}+a_{n+1}}{a_{n+1}+a_{n-1}} = \frac{a_{n+1}}{a_{n-1}} \quad \text{for } n \ge 2.$$
Prove that all terms of this sequence are positive integers and that a_{2000} is divisible by 2^{2000}.

2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,
$$f(x - f(y)) = 1 - x - y.$$

3. The diagonals of a cyclic quadrilateral $ABCD$ intersect at E. Let F and G be the midpoints of AB and CD respectively. Prove that the lines through E, F and G perpendicular to AD, BD and AC, respectively, intersect in a single point.

4. Three boxes with at least one marble in each are given. In each step we double the number of marbles in one of the boxes, taking the required number of boxes from one of the other two boxes. Is it always possible to have one of the boxes empty after several steps?