20-th All-Russian Mathematical Olympiad 1994

Fourth Round

Grade 9

First Day

1. One day, Rabbit was about to go for a meeting with Donkey, but Winnie the Pooh and Duck unexpectedly came to his home. Being well-bred, Rabbit offered the guests some refreshments. Pooh tied Duck's mouth by a napkin and ate 10 pots of honey and 22 cups of condensed milk alone, whereby he needed two minutes for each pot of honey and 1 minute for each cup of milk. Knowing that there was nothing sweet left in the house, Pooh released the Duck. Afflicted Rabbit observed that he wouldn't have been late for the meeting with Donkey if Pooh had shared the refreshments with Duck. Knowing that Duck needs 5 minutes for a pot of honey and 3 minutes for a cup of milk, he computed the time the guests would have needed to devastate his supplies. What is that time?

2. Cities A,B,C,D are positioned in such a way that A is closer to C than to D, and B is closer to C than to D. Prove that every point on the straight road from A to B is closer to C than to D.

3. Does there exist a quadratic trinomial $p(x)$ with integer coefficients such that, for every natural number n whose decimal representation consists of digits 1, $p(n)$ also consists only of digits 1?

4. On the world conference of parties of liars and truth-lovers there were 32 participants which were sitting in four rows with 8 chairs each. During a break each participant claimed that among his neighbors (by row or column) there are members of both parties. It is known that liars always lie, whereas truth-lovers always tell truth. What is the smallest number of liars at the conference for which this situation is possible?

Second Day

5. The equation $ax^5+bx^4+c=0$ has three distinct roots. Show that so does the equation $cx^5+bx+a=0$.

6. Point P is taken inside a right angle KLM. A circle $\mathscr{S}_1$ with center O_1 is tangent to the rays LK and LP of angle KLP at A and D respectively, and a circle $\mathscr{S}_2$ is tangent to the rays of angle MLP, touching LP at B. Suppose that O_1 lies on segment AB. Let the lines O_2D and KL meet at C. Prove that BC bisects the angle ABD.

7. Find all prime numbers p,q,r,s such that their sum is a prime number and p^2+qs and p^2+qr are squares of integers.

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8. There are 16 pupils in a class. Every month, the teacher divides the pupils into two groups. Find the smallest number of months after which it will be possible that every two pupils were in two different groups during at least one month.

Grade 10

First Day

1. We have seven equal pails with water, filled to one half, one third, one quarter, one fifth, one eighth, one ninth, and one tenth, respectively. We are allowed to pour water from one pail into another until the first pail empties or the second one fills to the brim. Can we obtain a pail that is filled to (a) one twelfth, (b) one sixth after several such steps?

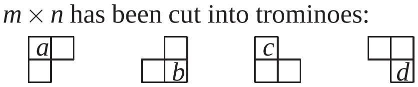
2. The equation $x^2 + ax + b = 0$ has two distinct real roots. Prove that the equation

$$x^4 + ax^3 + (b-2)x^2 - ax + 1 = 0$$

has four distinct real roots.

3. A circle with center O is inscribed in a quadrilateral $ABCD$ and touches its non-parallel sides BC and AD at E and F respectively. The lines AO and DO meet the segment EF at K and N respectively, and the lines BK and CN meet at M. Prove that the points O, K, M and N lie on a circle.

4. A rectangle of size $m \times n$ has been cut into trominoes:



Prove that the difference between the number of pieces a and the number of pieces b is divisible by 3.

Second Day

5. Find all prime numbers which can be written as a sum of two primes and as a difference of two primes.

6. Find the free coefficient of the polynomial $P(x)$ with integer coefficients, knowing that it is less than 1000 in absolute value and that $P(19) = P(94) = 1994$.

7. In a convex pentagon $ABCDE$ side AB is perpendicular to CD and side BC is perpendicular to DE. Prove that if $AB = AE = ED = 1$, then $BC + CD < 1$.

8. In the Flower-city there are n squares and m streets, where $m \geq n+1$. Each street connects two squares and does not pass through other squares. According to a tradition in the city, each street is named either blue or red. Every year, a square is selected and the names of all streets emanating from that square are changed. Show that the streets can be initially named in such a way that, no matter how the names will be changed, the streets will never all have the same name.

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Grade 11

First Day

1. Prove the inequality $\sin 2x + \cos x > 1$ for all x with $0 < x < \pi/3$.

2. It was noted that during one day in a town, each person made at most one phone call. Prove that the people in the town can be divided into three groups such that no two persons in the same group talked by phone that day.

3. A circle with center O is tangent to the sides AB, BC, AC of a triangle ABC at points E, F, D respectively. The lines AO and CO meet EF at points N and M. Prove that the circumcircle of triangle OMN and points O and D lie on a line.

4. On the vertices of a convex n-gon are put m stones, $m > n$. In each move we can choose two stones standing at the same vertex and move them to the two distinct adjacent vertices. After N moves the number of stones at each vertex was the same as at the beginning. Prove that N is divisible by n.

Second Day

5. *Problem 5 for Grade 10.*

6. Find all functions satisfying the equality
$$(x-1)f\left(\frac{x+1}{x-1}\right) - f(x) = x \quad \text{for all } x \neq 1.$$

7. Points A_1, B_1 and C_1 are taken on the respective edges SA, SB, SC of a regular triangular pyramid $SABC$ so that the planes $A_1B_1C_1$ and ABC are parallel. Let O be the center of the sphere passing through A, B, C_1 and S. Prove that the line SO is perpendicular to the plane A_1B_1C.

8. Points $A_1, A_2, \dots, A_n$ inside a circle and points $B_1, B_2, \dots, B_n$ on its boundary are positioned so that the segments $A_1B_1, A_2B_2, \dots, A_nB_n$ do not intersect. A bug can go from point A_i to A_j if the segment A_iA_j does not intersect any segment A_kB_k, $k \neq i, j$. Prove that the bug can go from any point A_p to any point A_q in a few steps.

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