16-th All-Russian Mathematical Olympiad 1990

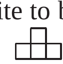
Final (Fourth) Round

Grade 9

First Day

1. Among 25 apparently equal coins 3 are fakes. All true coins have equal masses, and so do all fake coins, though these are lighter than the true ones. How to find six true coins by two measurings on a balance without weights?

2. Numbers a,b,c,d,p,q satisfy the relation $ab+cd=2pq$. Show that if $ac \geq p^2 > 0$, then $bd \leq q^2$.

3. An $n \times n$ chessboard ($n \geq 3$) is given. In each step it is allowed to change the colors (from white to black and vice-versa) in some four squares forming a figure congruent to . Is it possible to change the colors of all squares of the chessboard by several steps?

4. There are 1990 points inside a square with side length 12. Prove that there exists an equilateral triangle with side 11 which contains at least 498 of these points.

Second Day

5. Find all positive values of a for which both roots of the equation $a^2x^2+ax+1-7a^2=0$ are integers.

6. From an $n \times n$ square a 1×1 corner square has been cut off. What is the smallest number of triangles into which one can cut the obtained figure?

7. Circles S_1 and S_2 intersect at points A and B, with the center O of S_1 lying on S_2. A chord AC of S_1 intersects S_2 again at D. Prove that the segments OD and BC are perpendicular.

8. Among the sides and diagonals of a convex pentagon $ABCDE$ there are no parallel segments. The side AB is oriented towards the intersection point of lines AB and CE; the side BC is oriented towards the intersection of BC and DA; the sides CD, DE, EA are oriented analogously. Prove that there always exist two sides directed towards the same vertex.

Grade 10

First Day

1. Prove that for any positive numbers a,b,c the following inequality holds:
$$\sqrt{ab(a+b)}+\sqrt{bc(b+c)}+\sqrt{ca(c+a)} > \sqrt{(a+b)(b+c)(c+a)}.$$

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2. There are 100 points inside a circle, none at the center of the circle and no two at the same radius.

   (a) Prove that there exists a sector of the circle with the central angle of $2\pi/11$ containing exactly 10 of the given points.

   (b) Does there necessarily exist a sector of the circle with the central angle of $2\pi/11$ containing exactly 11 points?

3. Four circles are situated inside a convex quadrilateral such that each circle is tangent to two adjacent sides and two other circles. If the quadrilateral is circumscribed around a circle, prove that at least two of the four circles are congruent.

4. The triples (x_n, y_n, z_n), $n = 1, 2, \dots$ are constructed as follows:

$$x_1 = 2,\ \ y_1 = 4,\ \ z_1 = \tfrac{6}{7};$$
$$x_{n+1} = \frac{2x_n}{x_n^2 - 1},\ \ y_{n+1} = \frac{2y_n}{y_n^2 - 1},\ \ z_{n+1} = \frac{2z_n}{z_n^2 - 1}.$$

   (a) Prove that this process can be indefinitely continued.

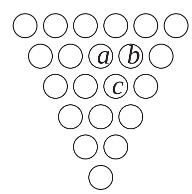
   (b) Will ever occur a triple (x_k, y_k, z_k) with $x_k + y_k + z_k = 0$?

Second Day

5. Several positive numbers are written such that the sum of their pairwise products equals 1. Show that it is possible to erase one of these numbers so that the sum of the remaining numbers is less than $\sqrt{2}$.

6. Circles S_1 and S_2 intersect at points A and B, where the center O of S_2 lies on S_1. A chord OC of S_1 intersects S_2 at D (between O and C). Prove that point D is the incenter of the triangle ABC.

7. Is it possible to write the natural numbers 1 through 21 in the cells on the picture, such that in each row but the first, each number equals the absolute difference of the numbers standing above it (for example, $c = |a - b|$)?



8. Let be given n distinct vectors with the following property: For some natural number p, the sum of any p of these vectors has the same length as the sum of the remaining $n - p$ vectors.

   (a) Prove that if $p \neq n/2$, then the sum of the given vectors is zero.

   (b) Is the sum of the vectors necessarily zero if $p = n/2$?

Grade 11

First Day

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1. *Problem 1 for Grade 9.*

2. Prove that in each triangle with sides a,b,c, corresponding angles A,B,C and perimeter p, the following inequality holds:

$$a\cos A + b\cos B + c\cos C \leq p.$$

3. Given 8 unit cubes, 24 of their faces are painted in white and the remaining 24 faces in black. Show that it is always possible to assemble these cubes into a cube of edge 2 on whose surface there are equally many white and black unit squares.

4. Let AB be a chord of a unit circle of length less than $\sqrt{2}$. A line l forms an angle of 45° with the line AB and does not meet the circle. Construct, by a ruler and a compass, point C on l for which the segments DE and AB are perpendicular, where E and D are the intersection points of segments CA and CB with the circle.

Second Day

5. Find all positive integers x,y satisfying $7^x - 3\cdot 2^y = 1$.

6. Points D and E are taken on the sides AB and BC respectively of a triangle ABC. Points K and M divide the segment DE into three equal parts. If the lines BK and BM intersect the side AC at T and P, prove that $TP \leq \frac{1}{3}AC$.

7. The basis of a pyramid $SABCDE$ is the regular pentagon $ABCDE$. The foot M of the altitude SM of the pyramid lies inside the pentagon $ABCDE$. Assume that the radii of the circumspheres of the tetrahedra $SMAB$, $SMBC$, $SMCD$ are equal. Does it follow that the pyramid is regular?

8. In each cell of a rectangular board there is a positive number. The sum of elements in each row is computed, and the product of these sums is denoted by P. Prove that, if the numbers in each column are rearranged in the increasing order from the top to the bottom, then the product P in the new board does not exceed its initial value.