# 32-nd All-Russian Mathematical Olympiad 2006

## Final Round – Pskov, April 21–26

**Grade 9**

*First Day*

1. Some pairs of centers of neighboring squares (by side) of a $15 \times 15$ board are connected by a segment so that a closed simple polygonal line symmetric with respect to one of the diagonals of the board is obtained. Prove that the length of the polygonal line does not exceed 200.

2. Show that there exist four integers $a,b,c,d$ whose absolute values are greater than $1,000,000$ such that
$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}=\frac{1}{abcd}.$$

3. Petya paints 2006 points on a circle in 17 colors. Then Kolya draws chords having the endpoints of the same color, so that no two chords have common points (including the endpoints). Thereby Kolya wants to draw as many chords as possible, while Petya wants to spoil his efforts. What largest number of chords can Kolya always draw?

4. A circle $\omega$ touches the circumcircle of a triangle $ABC$ at $A$, intersects side $AB$ at $K$, and intersects side $BC$. A tangent $CL$ to $\omega$ (with $L$ on $\omega$) is such that the segment $KL$ intersects side $BC$ at $T$. Prove that the length of $BT$ equals the length of the tangent from $B$ to $\omega$.

*Second Day*

5. Let $a_1 < a_2 < \cdots < a_{10}$ be natural numbers and let $b_k$ be the largest divisor of $a_k$ with $b_k < a_k$. Suppose that $b_1 > b_2 > \cdots > b_{10}$. Prove that $a_{10} > 500$.

6. Points $P,Q,R$ are taken on the sides $AB,BC,CA$ respectively of a triangle $ABC$ such that $AP = CQ$ and the quadrilateral $RPBQ$ is cyclic. The tangents to the circumcircle of triangle $ABC$ at $A$ and $C$ meet the respective lines $RP$ and $RQ$ at $X$ and $Y$. Prove that $RX = RY$.

7. A $100 \times 100$ square board has been cut into dominos, i.e. $1 \times 2$ rectangles. Two players play a game. A player in turn glues two squares sharing a side if there is a cut between them. A player loses if his move yields a connected figure, i.e. the entire board can be nailed to the table by just one nail. Who has a winning strategy - the first player or his opponent?

8. A quadratic trinomial $f(x) = x^2 + ax + b$ is given. The equation $f(f(x)) = 0$ has four real roots, two of which sum up to -1. Prove that $b \le -\frac{1}{4}$.



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D. Djukić, V. Janković, I. Matić, N. Petrović
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## Grade 10

*First Day*

1. *Problem 1 for Grade 9.*

2. Suppose that the sum of cubes of three consecutive positive integers is a perfect cube. Prove that among the three integers, the middle one is divisible by 4.

3. *Problem 3 for Grade 9.*

4. A circle $\omega$ touches the equal sides $AB$ and $AC$ of an isosceles triangle and intersects the side $BC$ at $K$ and $L$. The segment $AK$ meets $\omega$ again at $M$. Points $P$ and $Q$ are symmetric to $K$ with respect to $B$ and $C$, respectively. Prove that the circumcircle of triangle $PMQ$ is tangent to $\omega$.

*Second Day*

5. *Problem 5 for Grade 9.*

6. Points $K$ and $L$ are taken on the arcs $AB$ and $BC$ respectively of a triangle $ABC$ such that $KL$ is parallel to $AC$. Prove that the incenters of triangles $ABK$ and $CBL$ are equidistant from the midpoint of arc $ABC$.

7. *Problem 8 for Grade 9.*

8. A $3000 \times 3000$ square board has been cut into dominos. Show that the dominos can be painted in three colors in such a manner that the number of dominos of each color is the same and each domino has at most two neighboring dominos of the same color as itself. (Two dominos are neighboring if they contain two squares that are neighboring by side.)

## Grade 11

*First Day*

1. Prove that $\sin\sqrt{x} < \sqrt{\sin x}$ for $0 < x < \frac{\pi}{2}$.

2. The sum and product of two purely periodic decimal numbers is is a purely periodic decimal number of period $T$. Prove that the two initial numbers have periods not exceeding $T$.

3. Two players play a game on the $50 \cdot 70$ grid points of a $49 \times 69$ grid. A player in turn connects two of the grid points by a segment. No two segments can share an endpoint, but the segments may have common points. The segments are drawn as long as there are free grid points. If at the end the first player can direct the drawn segments so that the sum of the obtained vectors is zero, then he wins; otherwise the other player wins. Who wins if both players play reasonably?



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4. The bisectors $BB_1$ and $CC_1$ of a triangle $ABC$ meet at point $I$. The line $B_1C_1$ intersects the circumcircle of triangle $ABC$ at points $M$ and $N$. Prove that the circumradius of the triangle $MIN$ is twice as long as the circumradius of $\triangle ABC$.

*Second Day*

5. The sequences of positive numbers $(x_n)$ and $(y_n)$ satisfy

$$x_{n+2} = x_n + x_{n+1}^2, \quad y_{n+2} = y_n^2 + y_{n+1} \quad \text{for all } n \in \mathbb{N}.$$

Prove that if the numbers $x_1, x_2, y_1, y_2$ are all greater than 1, then $x_n > y_n$ for some $n$.

6. The incircle of face $ABC$ of a tetrahedron $SABC$ is tangent to $AB, BC$, $CA$ at $D, E, F$ respectively. Points $A', B', C'$ are taken on the respective segments $SA, SB, SC$ such that $AA' = AD$, $BB' = BE$, and $CC' = CF$. Point $S'$ on the circumsphere of the tetrahedron is diametrically opposite to $S$. Prove that if $SI$ is the altitude of the tetrahedron, then $S'$ is equidistant from points $A', B', C'$.

7. Suppose that the polynomial $(x+1)^n - 1$ is divisible by a polynomial $P(x) = x^k + c_{k-1}x^{k-1} + \cdots + c_1x + c_0$ of an even degree $k$ whose all coefficients $c_0, \dots, c_{k-1}$ are odd integers. Prove that $n$ is divisible by $k+1$.

8. There are a number of pioneers in a camp, each of which knows at least 50 and at most 100 others. Show that the pioneers can be given caps painted in 1331 colors in total in such a way that the caps of any two pioneers who know each other differ in at least 20 colors.



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