30-th All-Russian Mathematical Olympiad 2004

Final Round – Cheboksary, April 19–25

Grade 9

First Day – April 20

1. Every integer point of a coordinate plane is painted using one of the three colors, and each of the colors is used. Prove that there exists a right-angled triangle whose all vertices are of different colors. *(S. Berlov)*

2. A quadrilateral $ABCD$ is circumscribed about a circle. The bisectors of exterior angles at A and B, B and C, C and D, D and A meet at points K, L, M, N respectively. Let K_1, L_1, M_1, N_1 be the orthocenters of triangles ABK, BCL, CDM, DAN, respectively. Show that the quadrilateral $K_1L_1M_1N_1$ is a parallelogram. *(I. Emelyanov)*

3. There are 2004 boxes on a table, each of them containing a ball. We know that some of the balls are white and that their number is even. We are allowed to choose any two boxes and ask whether at least one of them contains a white ball. What is the minimum number of questions that guarantees us to be able to specify some two boxes containing white balls? *(The jury)*

4. If $x_1, x_2, \dots, x_n$ $(n > 3)$ are positive numbers whose product is 1, show that
$$\frac{1}{1+x_1+x_1x_2} + \frac{1}{1+x_2+x_2x_3} + \cdots + \frac{1}{1+x_n+x_nx_1} > 1.$$
(S. Berlov)

Second Day – April 21

5. Do there exist pairwise distinct natural numbers m, n, p, q such that $m+n = p+q$ and $\sqrt{m} + \sqrt[3]{n} = \sqrt{p} + \sqrt[3]{q} > 2004$? *(I. Bogdanov)*

6. There are 2004 telephones in the president's office, every two of which are connected by a cord of one of the four colors. There is at least one cord of each color. Is it always possible to select several telephones so that among the cords between them exactly three colors occur? *(O. Podlipskiy)*

7. The numbers from 1 to 100 are arranged around a circle in such a manner that every number is either greater than both of its neighboprs, or smaller than both of its neighbors. A pair of naighboring numbers is called *good* if the above condition is preserved when these two numbers are omitted. How many good pairs at least must be there? *(S. Berlov)*

8. Let O be the circumcenter of an acute-angled triangle ABC, T the circumcenter of $\triangle AOC$, and M the midpoint of AC. Points D on side AB

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and E on side BC are taken so that $\angle BDM = \angle BEM = \angle ABC$. Show that $BT \perp DE$.

(A. Smirnov)

Grade 10

First Day

1. *Problem 1 for Grade 9.*
2. *Problem 3 for Grade 9.*
3. The incircle of an inscribed and circumscribed quadrilateral $ABCD$ touches its sides AB, BC, CD, DA at K, L, M, N, respectively. The external bisectors of the angles at A and B meet at K', those of $\angle B$ and $\angle C$ meet at L', of $\angle C$ and $\angle D$ at M', and of $\angle D$ and $\angle A$ at N'. Prove that the lines KK', LL', MM', NN' pass through a single point.

   (S. Berlov, L. Emelyanov, A. Smirnov)
4. *Problem 4 for Grade 9.*

Second Day

5. A sequence od nonnegative rational numbers $(a_n)_{n=1}^{\infty}$ satisfies the relation $a_m + a_n = a_{mn}$ for all $m, n \in \mathbb{N}$. Prove that not all terms of this sequence are distinct.

   (A. Protopopov)
6. There are 1001 towns in a country, and any two of them are joined by a one-way road. At every town exactly 500 roads start. A republic containing 668 of the towns secedes. Prove that it is possible to travel between any two cities in this republic without leaving the republic.

   (D. Karpov, A. Smirnov)
7. A triangle T is contained in a convex, centrally symmetric polygon M. Let T' be the triangle symmetric to T with respect to a point P inside T. Prove that at least one of the vertices of T' lies inside the polygon M or on its boundary.

   (V. Dolnikov)
8. Does there exist a natural number $n > 10^{1000}$, not divisible by 10, with the property that one can exchange the places of some two distinct nonzero digits of n not affecting the set of its prime divisors?

   (Ye. Chernyshov, I. Bogdanov)

Grade 11

First Day

1. *Problem 1 for Grade 9.*

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2. Let I_A and I_B be the excenters of a triangle ABC corresponding to A and B respectively, and P be a point on a circumcircle Ω of the triangle. Prove that the midpoint of the segment whose endpoints are the circumcenters of the triangles I_ACP and I_BCP coincides with the center of Ω.

(A. Akopyan, L. Emelyanov)

3. Let $P(x)$ and $Q(x)$ be polynomials, and suppose that there is a polynomial $R(x,y)$ such that $P(x)-P(y)=R(x,y)(Q(x)-Q(y))$ for all x,y. Prove that there exists a polynomial $S(x)$ such that $P(x)=S(Q(x))$.

(A. Bystrikov)

4. The numbers $1,2,\dots,2004$ are written in the cells of a board with 9 rows and 2004 columns, and each of the numbers occurs exactly 9 times. Thereby no two numbers in the same column differ by more than 3. Find the smallest possible sum of the numbers in the first row.

(I. Bogdanov, G. Chelnokov)

Second Day

5. Let $M=\{x_1,\dots,x_{30}\}$ be a set of 30 distinct positive numbers and let A_n $(1\le n\le 30)$ be the sum of all products of n distinct elements of M. Show that if $A_{15}>A_{10}$, then $A_1>1$. *(V. Senderov)*

6. Prove that there is no finite set containing more than $2N$ pairwise non-collinear vectors in a plane ($N>3$) with the following properties:

   (i) For any N vectors from this set there are another $N-1$ vectors from the set such that the sum of all the $2N-1$ vectors is zero.

   (ii) For any N vectors from this set there are another N vectors from the set such that the sum of all the $2N$ vectors is zero. *(O. Podlipskiy)*

7. There are several cities in a country, and some of them are connected by two-way air routes served by k airliners. Every two routes served by the same airliner have a common endpoint. Prove that the towns can be partitioned into $k+2$ groups in such a way that no two towns in the same group are connected by a route. *(V. Dolnikov)*

8. A plane section of a rectangular parallelepiped is a hexagon. This hexagon is covered by a rectangle Π. Show that at least one face of the parallelepiped can be covered by the rectangle Π. *(S. Volchyonkov)*

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