26-th All-Russian Mathematical Olympiad 2000

Final Round – Kazan, April 14–15

Grade 9

First Day

1. Let a,b,c be distinct numbers such that the equations $x^2+ax+1=0$ and $x^2+bx+c=0$ have a common real root, and the equations $x^2+x+a=0$ and x^2+cx+b also have a common real root. Compute the sum $a+b+c$. *(N. Agakhanov)*

2. Tanya chose a natural number $X \leq 100$, and Sasha is trying to guess this number. He can select two natural numbers M and N less than 100 and ask about $\gcd(X+M,N)$. Show that Sasha can determine Tanya's number with at most seven questions. *(A. Golovanov)*

3. Let O be the center of the circumcircle ω of an acute-angled triangle ABC. A circle ω_1 with center K passes through A,O,C and intersects AB at M and BC at N. Point L is symmetric to K with respect to the line NM. Prove that $BL \perp AC$. *(M. Sonkin)*

4. Some pairs of cities in a certain country are connected by roads, at least three roads going out of each city. Prove that there exists a round path consisting of roads whose number is not divisible by 3. *(D. Karpov)*

Second Day

5. The sequence $a_1=1,a_2,a_3,\dots$ is defined as follows: if a_n-2 is a natural number not already occurring on the board, then $a_{n+1}=a_n-2$; otherwise $a_{n+1}=a_n+3$. Prove that every nonzero perfect square occurs in the sequence as the previous term increased by 3. *(N. Agakhanov)*

6. On some cells of a $2n \times 2n$ board are placed white and black markers (at most one marker on every cell). We first remove all black markers which are in the same column with a white marker, then remove all white markers which are in the same row with a black one. Prove that either the number of remaining white markers or that of remaining black markers does not exceed n^2. *(S. Berlov)*

7. Let E be a point on the median CD of a triangle ABC. The circle $\mathscr{S}_1$ passing through E and touching AB at A meets the side AC again at M. The circle $\mathscr{S}_2$ passing through E and touching AB at B meets the side BC at N. Prove that the circumcircle of $\triangle CMN$ is tangent to both $\mathscr{S}_1$ and $\mathscr{S}_2$. *(M. Sonkin)*

8. One hundred natural numbers whose gcd (greatest common divisor) is 1 are arranged around a circle. An allowed operation is to add to a number the gcd of its two neighbors. Prove that we can make all the numbers pairwise coprime in a finite number of moves.

(S. Berlov)

Grade 10

First Day

1. Evaluate the sum $\left[\frac{2^0}{3}\right]+\left[\frac{2^1}{3}\right]+\left[\frac{2^2}{3}\right]+\cdots+\left[\frac{2^{1000}}{3}\right]$. *(A. Golovanov)*

2. Let $-1 < x_1 < x_2 < \cdots < x_n < 1$ and $x_1^{13}+x_2^{13}+\cdots+x_n^{13} = x_1+x_2+\cdots+x_n$. Prove that if $y_1 < y_2 < \cdots < y_n$, then
$$x_1^{13}y_1+\cdots+x_n^{13}y_n < x_1y_1+x_2y_2+\cdots+x_ny_n.$$
(O. Musin)

3. In an acute scalene triangle ABC the bisector of the acute angle betweeen the altitudes AA_1 and CC_1 meets the sides AB and BC at P and Q respectively. The bisector of the angle B intersects the segment joining the orthocenter of ABC and the midpoint of AC at point R. Prove that P,B,Q,R lie on a circle. *(S. Berlov)*

4. We are given five equal-looking weights of pairwise distinct masses. For any three weights A,B,C, we can check by a measuring if $m(A) < m(B) < m(C)$, where $m(X)$ denotes the mass of a weight X (the answer is *yes* or *no*). Can we always arrange the masses of the weights in the increasing order with at most nine measurings? *(O. Podlipskiy)*

Second Day

5. Let M be a finite set of numbers, such that among any three of its elements there are two whose sum belongs to M. Find the greatest possible number of elements of M. *(E. Cherepanov)*

6. A perfect number, greater than 6, is divisible by 3. Prove that it is also divisible by 9. (A natural number is *perfect* if the sum of its proper divisors is equal to the number itself: e.g. $6 = 1+2+3$.) *(A. Hrabrov)*

7. Two circles are internally tangent at point N. Chords BA and BC of the larger circle touch the smaller circle at K and M respectively. Let Q and P be the midpoints of the arcs AB and BC respectively. The circumcircles of the triangles BQK and BPM meet again at B_1. Prove that BPB_1Q is a parallelogram. *(T. Emelyanov)*

8. Some paper squares of k distinct colors are placed on a rectangular table, with sides parallel to the sides of the table. Suppose that for any k squares of distinct colors, some two of them can be nailed on the table with only one nail. Prove that there is a color such that all squares of that color can

be nailed with $2k-2$ nails. *(V. Dolnikov)*

Grade 11

First Day

1. Determine all functions $f:\mathbb{R}\to\mathbb{R}$ such that for all real x,y,z,
$$f(x+y)+f(y+z)+f(z+x)\geq 3f(x+2y+3z).$$
(N. Agakhanov, O. Podlipskiy)

2. Prove that one can partition the set of natural numbers into 100 nonempty subsets such that among any three natural numbers a,b,c satisfying $a+99b=c$, there are two that belong to the same subset.
(F. Petrov, I. Bogdanov, S. Berlov, D. Djukić)

3. A convex pentagon $ABCDE$ is given in the coordinate plane with all vertices in lattice points. Prove that there must be at least one lattice point in the pentagon determined by the diagonals AC,BD,CE,DA,EB or on its boundary. *(V. Dolnikov, I. Bogdanov)*

4. Let be given a sequence of nonnegative integers $a_1,a_2,\dots,a_n$. For $k=1,2,\dots,n$, denote
$$m_k=\max_{1\leq l\leq k}\frac{a_{k-l+1}+a_{k-l+2}+\cdots+a_k}{l}.$$
Prove that for every $\alpha>0$ the number of values of k for which $m_k>\alpha$ is less than $\dfrac{a_1+a_2+\cdots+a_n}{\alpha}$. *(V. Dolnikov)*

Second Day

5. Prove the inequality $\sin^n(2x)+(\sin^n x-\cos^n x)^2\leq 1$. *(A. Hrabrov)*

6. A perfect number, greater than 28, is divisible by 7. Prove that it is also divisible by 49. *(A. Hrabrov)*

7. A quadrilateral $ABCD$ is circumscribed about a circle ω. The lines AB and CD meet at O. A circle ω_1 is tangent to side BC at K and to the extensions of sides AB and CD, and a circle ω_2 is tangent to side AD at L and to the extensions of sides AB and CD. Suppose that points O,K,L lie on a line. Prove that the midpoints of BC and AD and the center of ω also lie on a line. *(P. Kozhevnikov)*

8. The cells of a 100×100 board are painted in four colors in such a way that every row and every column contains exactly 25 cells of each color. Prove that there exist two rows and two columns, whose four intersection cells are all of distinct colors. *(S. Berlov)*