Romanian Team Selection Tests 1989

Selection Test for Balkan MO
Suceava, April 14

1. Let M denote the set of $m \times n$ matrices with entries in the set $\{0, 1, 2, 3, 4, \}$ such that in each row and each column the sum of elements is divisible by 5. Find the cardinality of set M. *(C. Năstăsescu)*

2. Let P be a point on a circle $\mathscr{C}$ and let φ be a given angle incomensurable with 2π. For each $n \in \mathbb{N}$, P_n denotes the image of P under the rotation about the center O of $\mathscr{C}$ by the angle $\alpha_n = n\varphi$. Prove that the set $\mathscr{M} = \{P_n \mid n \geq 0\}$ is dense in $\mathscr{C}$. *(V. Barbu)*

3. Let $ABCD$ be a parallelogram and M, N be points in the plane such that $C \in (AM)$ and $D \in (BN)$. Lines NA, NC meet lines MB, MD at points E, F, G, H. Show that points E, F, G, H lie on a circle if and only if $ABCD$ is a rhombus.

4. A family of finite sets $\{A_1, A_2, \dots, A_m\}$ is called *equipartitionable* if there is a function $\varphi : \bigcup_{i=1}^{m} A_i \to \{-1, 1\}$ such that $\sum_{x \in A_i} \varphi(x) = 0$ for every $i = 1, \dots, m$. Let $f(n)$ denote the smallest possible number of n-element sets which form a non-equipartitionable family. Prove that
   (a) $f(4k+2) = 3$ for each nonnegative integer k;
   (b) $f(2n) \leq 1 + md(n)$, where $md(n)$ denotes the least positive non-divisor of n. *(I. Tomescu)*

5. A *laticial cycle* of length n is a sequence of lattice points (x_k, y_k), $k = 0, 1, \dots, n$, such that $(x_0, y_0) = (x_n, y_n) = (0, 0)$ and $|x_{k+1} - x_k| + |y_{k+1} - y_k| = 1$ for each k. Prove that for all n, the number of latticial cycles of length n is a perfect square. *(G. Eckstein)*

First Test for IMO
Iaşi, June 12

1. Let the sequence (a_n) be defined by $a_n = n^6 + 5n^4 - 12n^2 - 36$, $n \geq 2$.
   (a) Prove that any prime number divides some term in this sequence.
   (b) Prove that there is a positive integer not dividing any term in the sequence.
   (c) Determine the least $n \geq 2$ for which $1989 \mid a_n$. *(E. Popa)*

2. Find all monic polynomials $P(x), Q(x)$ with integer coefficients such that $Q(0) = 0$ and
$$P(Q(x)) = (x-1)(x-2)\cdots(x-15).$$

(M. Dădârlat, G. Eckstein)

3. Find all pairs (m,n) of integers with $m > 1$, $n \geq 3$ with the following property: if an n-gon can be partitioned into m isosceles triangles, then the n-gon has two congruent sides. *(D. Tătaru)*

4. Let r,n be positive integers. For a set A, let $\binom{A}{r}$ denote the family of all r-element subsets of A. Prove that if A is infinite and $f : \binom{A}{r} \to \{1,2,\dots,n\}$ is any function, then there exists an infinite subset B of A such that $f(X) = f(Y)$ for all $X,Y \in \binom{B}{r}$. *(I. Tomescu)*

Second Test for IMO
Iaşi, June 13

1. Let F be the set of all functions $f : \mathbb{N} \to \mathbb{N}$ which satisfy
$$f(f(x)) - 2f(x) + x = 0 \quad \text{for all } x \in \mathbb{N}.$$
Determine the set $A = \{f(1989) \mid f \in F\}$. *(G. Eckstein)*

2. Let a,b,c be coprime nonzero integers. Prove that for any coprime integers u,v,w with $au+bv+cw = 0$ there exist integers m,n,p such that
$$a = nw - pv, \quad b = pu - mw, \quad c = mv - nu.$$
(O. Stănăşilă)

3. (a) Find the point M in the plane of triangle ABC for which the sum $MA + MB + MC$ is minimal.
   (b) Given a parallelogram $ABCD$ whose angles do not exceed 120°, determine
$$\min\{MA+MB+NC+ND+MN \mid M,N \text{ are in the plane } ABCD\}$$
   in terms of the sides and angles of the parallelogram. *(S. Aniţa)*

4. Let A,B,C be variable points on edges OX,OY,OZ of a trihedral angle $OXYZ$, respectively. Let $OA = a$, $OB = b$, $OC = c$ and R be the radius of the circumsphere S of $OABC$. Prove that if points A,B,C vary so that $a+b+c = R+l$, then the sphere S remains tangent to a fixed sphere. *(D. Tătaru)*

Third Test for IMO
Iaşi, June 14

1. Prove that for eny $n \in \mathbb{N}$, $\sqrt{1+\sqrt{2+\cdots+\sqrt{n}}} < 2$.

(G. Eckstein)

2. The sequence (a_n) is defined by $a_1 = a_2 = 1$, $a_3 = 199$ and

$$a_{n+1} = \frac{1989 + a_n a_{n-1}}{a_{n-2}} \quad \text{for all } n \geq 3.$$

   Prove that all terms of the sequence are positive integers. *(E. Popa)*

3. Let $\mathscr{F}$ be the boundary and M, N be any interior points of a triangle ABC. Consider the function $f_{M,N} : \mathscr{F} \to \mathbb{R}$ defined by $f_{M,N}(P) = MP^2 + NP^2$ and let $\eta_{M,N}$ be the number of points P for which $f_{M,N}$ attains its minimum.

   (a) Prove that $1 \leq \eta_{M,N} \leq 3$.

   (b) If M is fixed, find the locus of N for which $\eta_{M,N} > 1$.

   (c) Prove that the locus of M for which there are points N such that $\eta_{M,N} = 3$ is the interior of a tangent hexagon. *(D. Brânzei)*