49-th Romanian Mathematical Olympiad 1998

Final Round

Vaslui, March 24–30, 1998

7-th Form

1. Let n be a positive integer and $x_1, x_2, \dots, x_n$ be integers satisfying
$$x_1^2 + x_2^2 + \cdots + x_n^2 + n^3 \le (2n-1)(x_1 + x_2 + \cdots + x_n) + n^2.$$
   (a) Show that $x_1, x_2, \dots, x_n$ are nonnegative.
   (b) Show that $x_1 + x_2 + \cdots + x_n + n + 1$ is not a perfect square.
(S. Smarandache)

2. Show that there is no positive integer n such that $n + k^2$ is a perfect square for at least n positive integers k. *(V. Zidaru)*

3. In the exterior of triangle ABC with $\angle B > 45^\circ$ and $\angle C > 45^\circ$ the right isosceles triangles ACM, ABN with the right angles at A are constructed. Also, the right isosceles triangle BCP with $\angle P = 90^\circ$ is constructed in the interior of $\triangle ABC$. Show that MNP is a right isosceles triangle.
(B. Enescu)

4. Let E be the point on the diagonal BD of a rectangle $ABCD$ such that $\angle DAE = 15^\circ$, and let F be the foot of theperpendicular from E to BD. Given that $EF = AB/2$ and $AD = a$, find $\angle EAC$ and the length of segment EC. *(S. Peligrad)*

8-th Form

1. For a real number a, let $A = \{(x,y) \mid x,y \in \mathbb{R},\ x+y=a\}$ and $A = \{(x,y) \mid x,y \in \mathbb{R},\ x^3+y^3 < a\}$. Find all values of a for which A and B are disjoint. *(R. Ilie)*

2. Let $P(X) = a_{1998}X^{1998} + \cdots + a_1X + a_0$ be a polynomial with real coefficients such that $P(0) \ne P(-1)$, and let $Q(X) = b_{1998}X^{1998} + \cdots + b_1X + b_0$ be the polynomial given by $b_k = aa_k + b$ for all k, where a and b are given real numbers. Show that if $Q(0) = Q(-1) \ne 0$, then $Q(X)$ has no real roots. *(M. Fianu, Şt. Alexe)*

3. In a trapezoid $ABCD$ with $AB \parallel CD$ and $\angle A = 90^\circ$ we have $AD = DC = a$ and $AB = 2a$. Let E and F respectively be the points on the perpendiculars at C and D to the plane of the trapezoid, on the same side of the plane, such that $CE = 2a$ and $DF = a$. Compute the distance from B to the plane AEF and the angle between AF and BE.
(R. Popovici, N. Solomon)

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4. Let $ABCD$ be an arbitrary tetrahedron. The bisectors of angles $\angle BDC$, $\angle CDA$, $\angle ADB$ intersect BC, CA, AB in points M, N, P, respectively.

   (a) Show that the planes ADM, BDN and CDP have a common line d.

   (b) Let A', B', C' be points on rays AD, BD, CD respectively, so that $AA' = BB' = CC'$ and let G and G' be the centroids of triangles ABC and $A'B'C'$. Prove that the lines GG' and d are either parallel or identical. *(M. Miculiţa)*

9-th Form

1. Let $f(x) = ax^2 + bx + c$, where a, b, c are integers. Find a, b, c so that $f(f(1)) = f(f(2)) = f(f(3))$. *(C. Mortici, M. Chiriţă)*

2. If $ABCD$ is a cyclic (convex) quadrilateral, prove that
$$|AC - BD| \leq |AB - CD|.$$
   When does equality hold? *(D. Miheţ)*

3. For integers a, b, find all rational roots (if any) of the equation
$$abx^2 + (a^2 + b^2)x + 1 = 0.$$
   (D. Popescu)

4. Let $A_1A_2 \dots A_n$ be a regular polygon with $n > 4$. Let T be the intersection of lines A_1A_2 and $A_{n-1}A_n$ and M be any interior point of the triangle A_1A_nT. Show that the equality
$$\sum_{i=1}^{n-1} \frac{\sin^2 \angle A_iMA_{i+1}}{d(M, A_iA_{i+1})} = \frac{\sin^2 \angle A_1MA_n}{d(M, A_1A_n)},$$
   where $d(X, l)$ denotes the distance from point X to line l, holds if and only if M lies on the circumcircle of the polygon. *(D. Brânzei)*

10-th Form

1. Let $M = \{1, 2, \dots, n\}$, where $n \geq 2$ is an integer. For every $k = 1, 2, \dots, n-1$ we define $x_k = \frac{1}{n+1} \sum (\min A + \max A)$, where the sum goes over all k-element subsets A of M. Show that $x_1, \dots, x_{n-1}$ are integers, not all divisible by 4. *(V. Zidaru)*

2. Let $a \geq 1$ be a real number. Suppose z is a complex number such that $|z + a| \leq a$ and $|z^2 + a| \leq a$. Prove that $|z| \leq a$. *(D. Şerbănescu)*

3. Let A', B', C' be arbitrary points on edges DA, DB, DC respectively of a tetrahedron $ABCD$. Let points $P_a, P_b, P_c, P'_a, P'_b, P'_c$ on BC, CA, AB, $B'C', C'A', A'B'$, respectively, be defined by
$$\frac{P_bC}{P_bA} = \frac{P'_bC'}{P'_bA'} = \frac{CC'}{AA'}, \quad \frac{P_cA}{P_cB} = \frac{P'_cA'}{P'_cB'} = \frac{AA'}{BB'}, \quad \frac{P_aB}{P_aC} = \frac{P'_aB'}{P'_aC'} = \frac{BB'}{CC'}.$$

(a) Prove that the lines AP_a, BP_b, CP_c have a common point P and that the lines $A'P'_a, B'P'_b, C'P'_c$ have a common point P'.

(b) Prove that $\dfrac{PC}{PP_c} = \dfrac{P'C'}{P'P'_c}$.

(c) Prove that, as A', B', C' vary, the line PP' is always parallel to a fixed line. *(M. Miculiţa)*

4. Let $x_1 < x_2 < \cdots < x_n$ be positive integers, where $n \geq 2$. Define

$$s_k = \sum_A \frac{1}{\prod_{a\in A} a} \qquad \text{for } k = 1,2,\dots,n,$$

where the sum is taken over all nonempty subsets A of $\{x_1, x_2, \dots, x_k\}$. Prove that if s_n and s_{n-1} are positive integers, then s_k is a positive integer for each k. *(D. Mihet)*

11-th Form

1. The nonzero 2×2 matrices $A_0, A_1, \dots, A_n$ with real entries satisfy $A_0 \neq aI$ for any real a and $A_0A_k = A_kA_0$ for all k (where I denotes the identity 2×2 matrix). Prove that

(a) $\det\left(\sum_{k=1}^{n} A_k^2\right) \geq 0$;

(b) if $\det\left(\sum_{k=1}^{n} A_k^2\right) = 0$ and $A_2 \neq aA_1$ for any real a, then $\sum_{k=1}^{n} A_k^2 = 0$.

(V. Pop)

2. Let (a_n) be a sequence of real numbers such that the sequence $x_n = \sum_{k=1}^{n} a_k^2$ is convergent and the sequence $y_n = \sum_{k=1}^{n} a_k$ is unbounded. Prove that the sequence $b_n = y_n - [y_n]$ is divergent. *(B. Enescu)*

3. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function for which the inequality

$$f'(x) \leq f'\left(x + \frac{1}{n}\right)$$

holds for every $x \in \mathbb{R}$ and every $n \in \mathbb{N}$. Prove that f is continuously differentiable. *(M. Piticari)*

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that for every real numbers $a < b$ there exist $c_1 \leq c_2$ in the interval $[a,b]$ with

$$f(c_1) = \min_{a\leq x\leq b} f(x) \qquad \text{and} \qquad f(c_2) = \max_{a\leq x\leq b} f(x).$$

Show that the function f is increasing.

(C. Mortici)

12-th Form

1. Let a,b be positive real numbers such that $a+b<1$ and let $f:[0,\infty)\to[0,\infty)$ be an increasing function such that for every $x\geq 0$,

$$\int_0^x f(t)dt = \int_0^{ax} f(t)dt + \int_0^{bx} f(t)dt.$$

Prove that $f(x)=0$ for all $x\geq 0$. *(M. Piticari)*

2. (a) For a prime number p, let $G_p = \bigcup_{n\in\mathbb{N}} \{z\in\mathbb{C} \mid z^{p^n}=1\}$. Show that $(G_p,\cdot)$ is a subgroup of $\mathbb{C}^*$.

   (b) Let H be an infinite subgroup $\mathbb{C}^*$. Prove that every proper subgroup of H is infinite if and only if $H=G_p$ for some prime p.

(∗∗∗)

3. A ring A is called *boolean* if $x^2=x$ for each $x\in A$. Prove that:

   (a) One can define a structure of boolean ring on an n-element set ($n\geq 2$) if and only if $n=2^k$ for some $k\in\mathbb{N}$.

   (b) It is possible to define a structure of boolean ring on the set $\mathbb{N}$.

(M. Andronache, S. Dăscălescu, I. Savu)

4. Let $k\subseteq\mathbb{C}$ be a field such that

   (i) k has exactly two endomorphisms f and g;

   (ii) if $f(x)=g(x)$ then $x\in\mathbb{Q}$.

Prove that there exists a square-free integer $d>1$ such that $k=\mathbb{Q}\left[\sqrt{d}\right]$.
(M. Ţena)

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