58-th Romanian National Mathematical Olympiad 2007

Final Round

Piteşti, March 29, 2007

7-th Form

1. If the side lengths a, b, c of a triangle satisfy $a+b-c=2$ and $2ab-c^2=4$, show that the triangle is equilateral.

2. Consider a triangle ABC with a right angle at A and $AC = 2AB$. Let P and Q be the midpoints of AC and AB, respectively, and let M and N be two points on side BC such that $BM = CN = x$, with $2x < BC$. Express x in terms of AB if the area of $MNPQ$ is half the area of ABC.

3. Consider a triangle ABC with a right angle at A and $AB < AC$. Point D on the side AC is such that $\angle ACB = \angle ABD$. Let DE be the altitude in $\triangle BCD$. If $AC = BD + DE$, what are the angles of triangle ABC?

4. If $m > 1$ and n are integers such that $2^{2m+1} \geq n^2$, show that $2^{2m+1} \geq n^2 + 7$.

8-th Form

1. Prove that the number $10^{1}0$ cannot be written as a product of two positive integers all of whose decimal digits are nonzero.

2. Assume 2007 offices are assigned 6018 desks, where each office is assigned at least one desk. All desks from any one office may be moved from that office to other offices so that each office, apart from the one we have emptied, has the same number of desks. What are the possible desk ascriptions?

3. (a) If all sides of a triangle ABC have length less than 2, show that the length of the altitude from A is less than $\sqrt{4 - \frac{BC^2}{4}}$.

   (b) Show that the volume of a tetrahedron with five edges shorter than 2 is less than 1.

4. Let $ABCD$ be a tetrahedron and M a point in space such that $MA^2 + MB^2 + CD^2 = MB^2 + MC^2 + DA^2 = MC^2 + MD^2 + AB^2 = MD^2 + MA^2 + BC^2$. Show that M lies on the common perpendicular to the lines AC and BD.

www.imo.org.yu

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imo.org.yu

9-th Form

1. Let $a_1, \dots, a_n$ be positive integers and $a_{n+1} = a_1$. Suppose the polynomial
$$P(x) = x^2 - \left(\sum_{i=1}^{n} a_i^2 + 1 \right) x + \sum_{i=1}^{n} a_i a_{i+1}$$
has an integer root. Show that if n is a perfect square, then so are both roots of $P(x)$.

2. Let M be a point in the plane of an acute-angled triangle ABC. Show that
$$\frac{a}{MA}\overrightarrow{MA} + \frac{b}{MB}\overrightarrow{MB} + \frac{c}{MC}\overrightarrow{MC} = 0$$
if and only if M is the orthocenter of ABC.

3. The plane is partitioned into bands of width 1 by parallel lines, and each band is colored white of black. Show that one can always place an equilateral triangle of side 100 such that its vertices have the same color.

4. For $f : X \to X$, denote $f_0(X) = X$ and $f_{n+1}(X) = f(f_n(X))$ for $n \in \mathbb{N}_0$. Denote
$$f_\infty(X) = \bigcap_{n=0}^{\infty} f_n(X).$$
Prove that, if X is finite, then $f(f_\infty(X)) = f_\infty(X)$. Does the result still hold when X is infinite?

10-th Form

1. Let n be a positive integer. Prove that the equation $z^n + z + 1 = 0$ has a complex solution of modulus 1 if and only if $n = 3m + 2$ for some integer $m > 0$.

2. Solve the equation $2^{x^2+x} + \log_2 x = 2^{x+1}$ in the set of real numbers.

3. For which integers $n \geq 2$ is $(n-1)^{n^{n+1}} + (n+1)^{n^{n-1}}$ divisible by n^n?

4. (a) Let S be a finite set of numbers and let $S + S = \{x + y \mid x, y \in S\}$. Show that $|S + S| \leq \frac{|S|^2 + |S|}{2}$.

   (b) Given a positive integer m, let $C(m)$ be the greatest positive integer k such that, for some set S of m integers, every integer from 1 to k belongs to S or is a sum of two (not necessarily distinct) elements of S. For instance, $C(3) = 8$ with $S = \{1, 3, 4\}$. Show that $\frac{m(m+6)}{4} \leq C(m) \leq \frac{m(m+3)}{2}$.

11-th Form

1. If A and B are 2×2 matrices with real entries satisfying $A^2 + B^2 = AB$, prove that $(AB - BA)^2 = 0$.

2. Given two real numbers $a < b$ in the image of a continuous real function f on $\mathbb{R}$, prove that the closed interval $[a, b]$ is the image under f of some interval $I \subset \mathbb{R}$.

3. Given an integer $n \geq 2$, let Σ^{n-1} be the set of all n-tuples $x = (x_1, \dots, x_n)$ of real numbers with $|x_1| + \cdots + |x_n| = 1$. Determine the $n \times n$ matrices A with real entries such that $xA \in \Sigma^{n-1}$ for all $x \in \Sigma^{n-1}$.

4. A *P-function* is a differential function $f : \mathbb{R} \to \mathbb{R}$ with a continuous derivative f' on $\mathbb{R}$ such that $f(x + f'(x)) = f(x)$ for all $x \in \mathbb{R}$.

   (a) Prove that the derivative of a P-function has at least one zero.

   (b) Provide an example of a non-constant P-function.

   (c) Prove that a P-function whose derivative has at least two distinct zeros is constant.

12-th Form

1. Let $\mathcal{C}$ be the class of all differentiable functions $f : [0, 1] \to \mathbb{R}$ with a continuous derivative f' on $[0, 1]$ and $f(0) = 0$, $f(1) = 1$. Find the minimum value of the integral
$$\int_0^1 \sqrt{1 + x^2}(f'(x))^2 dx$$
when f runs over all of $\mathcal{C}$, and find all functions in $\mathcal{C}$ that achieve this minimum value.

2. Let $f : [0, 1] \to \mathbb{R}^+$ be a continuous function.

   (a) Given a positive integer n, prove that there is a unique subdivision $0 = a_0 < a_1 < \cdots < a_n = 1$ of $[0, 1]$ such that
$$\int_{a_k}^{a_{k+1}} f(x)dx = \frac{1}{n} \int_0^1 f(x)dx, \quad k = 0, \dots, n-1.$$

   (b) For each positive integer n, define $\bar{a}_n = \frac{a_1 + \cdots + a_n}{n}$, where $0 = a_0 < a_1 < \cdots < a_n = 1$ is the subdivision from part (a). Prove that the sequence $(\bar{a}_n)_{n \geq 1}$ is convergent and evaluate its limit.

3. Given a positive integer n, find the rings R with the property that $x^{2^n+1} = 1$ for all $x \in R \setminus \{0\}$.

4. For an integer $n \geq 3$, let G be a subgroup of the symmetric group S_n generated by $n - 2$ transpositions. Prove that, for each $i = 1, \dots, n$, the set $\{\sigma(i) \mid \sigma \in G\}$ has at most $n - 1$ elements.