50-th Polish Mathematical Olympiad 1998/99

Third Round

April 14–15, 1999

First Day

1. Point D is taken on the side BC of a triangle ABC such that $AD > BC$. Let E be a point on the side AC such that $\frac{AE}{EC} = \frac{BD}{AD-BC}$. Show that $AD > BE$.

2. Let $0 < a_1 < a_2 < \cdots < a_{101} < 5050$ be integers. Prove that there exist four different numbers a_k, a_l, a_m, a_n such that $a_k + a_l - a_m - a_n$ is divisible by 5050.

3. Let $S(x)$ denote the sum of digits of x. Show that there exist positive integers $n_1 < n_2 < \cdots < n_{50}$ such that

$$n_1 + S(n_1) = n_2 + S(n_2) = \cdots = n_{50} + S(n_{50}).$$

Second Day

4. Find all integers $n \geq 2$ for which the following system has a solution in integers:

$$\begin{array}{rcl} x_1^2 + x_2^2 + 50 & = & 16x_1 + 12x_2, \\ x_2^2 + x_3^2 + 50 & = & 16x_2 + 12x_3, \\ \cdots\cdots\cdots & & \cdots\cdots\cdots \\ x_n^2 + x_1^2 + 50 & = & 16x_n + 12x_1. \end{array}$$

5. If a_i, b_i $(i = 1, 2, \dots, n)$ are integers, prove that

$$\sum_{1 \le i < j \le n} (|a_i - a_j| + |b_i - b_j|) \le \sum_{1 \le i < j \le n} |a_i - b_j|.$$

6. A convex hexagon $ABCDEF$ satisfies

$$\angle A + \angle C + \angle E = 360^\circ \quad \text{and} \quad \frac{AB}{BC} \cdot \frac{CD}{DE} \cdot \frac{EF}{FA} = 1.$$

Prove that $\frac{AB}{BF} \cdot \frac{FD}{DE} \cdot \frac{EC}{CA} = 1$.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com