48-th Polish Mathematical Olympiad 1996/97

Third Round

April 4–5, 1997

First Day

1. Positive integers $x_1, x_2, \dots, x_7$ satisfy $x_{n+3} = x_{n+2}(x_{n+1} + x_n)$ for $n = 1,2,3,4$. If $x_6 = 144$, find x_7.

2. Find all triples (x,y,z) of real numbers satisfying

$$3(x^2+y^2+z^2) = 1,$$
$$x^2y^2+y^2z^2+z^2x^2 = xyz(x+y+z)^3.$$

3. The medians of the faces ABD, ACD, BCD of a tetrahedron $ABCD$ taken from D make equal angles with the edges they were led to. Prove that the area of each of the faces ABD, ACD, BCD is less than the sum of the areas of the remaining two.

Second Day

4. Consider the sequence given by $a_1 = 0$ and $a_n = a_{[n/2]} + (-1)^{\frac{n(n+1)}{2}}$ for $n > 1$. For each integer $k \geq 0$, find the number of indices n with $2^k \leq n < 2^{k+1}$ such that $a_n = 0$.

5. A convex pentagon $ABCDE$ with $DC = DE$ and $\angle DCB = \angle DEA = 90^\circ$ is given. Let F be a point on the segment AB such that $AF : BF = AE : BC$. Prove that

$$\angle FCE = \angle ADE \quad \text{and} \quad \angle FEC = \angle BDC.$$

6. Let be given n distinct points on a circle of radius 1. Let q be the number of the segments with endpoints in the given points whose length is greater than $\sqrt{2}$. Prove that $3q \leq n^2$.

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