47-th Polish Mathematical Olympiad 1995/96

Third Round

March 29–30, 1996

First Day

1. Determine all positive integers n and real numbers r such that the polynomial $2x^2+2x+1$ divides $(x+1)^n-r$.

2. Let P be an interior point of a triangle ABC such that $\angle PBC = \angle PCA < \angle PAB$. The line BP intersects the circumcircle of ABC again at point E. The circumcircle of triangle APE intersects CE again at F. Prove that $APEF$ is a convex quadrilateral and that the ratio of its area to the area of the triangle ABP does not depend on the choice of P.

3. Let $a_1, a_2, \dots, a_n$ be positive numbers with the sum 1.

   (a) Prove that for any positive $x_1, x_2, \dots, x_n$ with the sum 1 it holds that
   $$2\sum_{i<j} x_i x_j \le \frac{n-2}{n-1} + \sum_{i=1}^{n} \frac{a_i x_i^2}{1-a_i}.$$

   (b) In the above inequality, determine all cases of equality.

Second Day

4. Suppose that a tetrahedron $ABCD$ is such that $\angle BAC = \angle ACD$ and $\angle CDB = \angle DBA$. Prove that $AB = CD$.

5. For a natural number k, we denote by $p(k)$ the least prime number that does not divide k. Let us define $q(n)$ as the product of all primes smaller than $p(k)$ if $p(k) > 2$, and as 1 otherwise. The sequence (x_n) is given by $x_0 = 1$ and
   $$x_{n+1} = \frac{x_n p(x_n)}{q(x_n)} \qquad \text{for } n = 0, 1, 2, \dots$$
   Determine all positive integers n with $x_n = 111111$.

6. Consider the collection of all permutations f of the set $\{1, 2, \dots, n\}$ which satisfy $f(i) \ge i-1$ for all i. Let p_n be the probability that a randomly chosen permutation from this collection also satisfies $f(i) \le i+1$ for all i. Determine all n for which $p_n > 1/3$.

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