46-th Polish Mathematical Olympiad 1994/95

Third Round

Gdynia, March 31 – April 1, 1995

First Day

1. Find the number of subsets of $\{1,2,\dots,2n\}$ in which the equation $x+y=2n+1$ has no solutions.

2. A convex pentagon is cut by its diagonals into a pentagon and ten triangles. What is the largest number of the obtained triangles which may have the same area?

3. Let $p>3$ be a prime and let $q=p^3$. The sequence (a_n) is defined by

$$a_n=\begin{cases} n & \text{for} \quad n=0,1,2,\dots,p-1; \\ a_{n-1}+a_{n-p} & \text{for} \quad n\geq p. \end{cases}$$

   Determine the remainder when a_q is divided by p.

Second Day

4. Let $x_1,x_2,\dots,x_n$ be positive numbers with the harmonic mean equal to 1. Find the smallest possible value of

$$x_1+\frac{x_2^2}{2}+\frac{x_3^3}{3}+\cdots+\frac{x_n^n}{n}.$$

5. An urn contains n sheets of papers labelled $1,2,\dots,n$. We draw the sheets one by one without putting them back into the urn until we obtain a sheet with a number divisible by k. For a fixed n, determine all values of k for which the expected value of the number of drawings is equal to k.

6. Three rays k,l,m in the space with a common endpoint P and a point $A\neq P$ on k are given. Prove that there exists exactly one pair of points $B\in l$ and $C\in m$ such that

$$PA+AB=PC+CB \qquad \text{and} \qquad PB+BC=PA+AC.$$

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