44-th Polish Mathematical Olympiad 1992/93

Third Round

Warszawa, April 4–5, 1993

First Day

1. Find all rational solutions x, y, z, t, w to the following system:

$$\begin{aligned} 2xy &= t^2 - w^2 + z^2, \\ 2xz &= t^2 - y^2 + w^2, \\ 2yz &= t^2 - w^2 + x^2. \end{aligned}$$

2. A circle k with center O is inscribed in a nonisosceles trapezoid $ABCD$ with the longer base AB. Let M be the midpoint of AB. The line CD touches k at point E and intersects the line OM at point F. Prove that $DE = FC$ if and only if $AB = 2CD$.

3. Let $g(k)$ denote the greatest odd divisor of a positive integer k. We set

$$f(k) = \begin{cases} k/2 + k/g(k) & \text{for } k \text{ even;} \\ 2^{(k+1)/2} & \text{for } k \text{ odd.} \end{cases}$$

and define the sequence x_n by $x_1 = 1$ and $x_{n+1} = f(x_n)$ for $n \in \mathbb{N}$. Show that the number 800 appears in the sequence exactly once, and determine n for which $x_n = 800$.

Second Day

4. Let be given a convex polyhedron whose all faces are triangular. The vertices of the polyhedron are colored using three colors. Prove that the number of faces with vertices in all the three colors is even.

5. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the following conditions:

$$f(-x) = -f(x) \quad \text{and} \quad f(x+1) = f(x) + 1 \quad \text{for } x \in \mathbb{R};$$

$$f\left(\frac{1}{x}\right) = \frac{f(x)}{x^2} \quad \text{for } x \neq 0.$$

6. Find out whether it is possible to determine the volume of a tetrahedron knowing the areas of its faces and its circumradius.

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