40-th Polish Mathematical Olympiad 1988/89

Third Round

First Day

1. An even number of people are participating in a round table conference. After lunch break the participants change seats. Show that some two persons are separated by the same number of persons as they were before break.

2. Circles K_1, K_2, K_3 are given in the plane such that K_2 and K_3 are tangent at P, K_3 and K_1 at Q, and K_1 and K_2 at R.Lines PQ and PR cut K_1 again at S and T respectively. Lines SR and TQ cut K_2 and K_3 again at U and V. Prove that points P, U, V are collinear.

3. The edges of a cube are numbered 1 through 12.

   (a) Prove that for any such numbering there exist at least eight triples of integers (i, j, k) with $1 \leq i < j < k \leq 12$ such that the edges assigned numbers i, j, k are consecutive segments of a polygonal line.

   (b) Give an example of a numbering for which there are exactly eight such triples.

Second Day

4. Let be given positive integers n and k. Consider a chain of sets $A_0, A_1, \dots, A_k$ in which $A_0 = \{1, \dots, n\}$ and, for each i, A_i is a randomly chosen subset of A_{i-1} (all choices are equiprobable). Show that the expected cardinality of A_k is $n/2^k$.

5. The pairwise equal circles of equal radius a lie on a hemisphere of radius r. Compute the radius of a fourth circle on the same sphere which is tangent to the three given circles.

6. Let a, b, c, d be positive numbers. Prove the inequality

$$\sqrt{\frac{ab + ac + ad + bc + bd + cd}{6}} \geq \sqrt[3]{\frac{abc + abd + acd + bcd}{4}}.$$

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