32-nd Polish Mathematical Olympiad 1980/81

Third Round

First Day

1. Two intersecting lines a and b are given in a plane. Consider all pairs of orthogonal planes α,β such that $a\subset\alpha$ and $b\subset\beta$. Prove that there is a circle such that every its point lies on the line $\alpha\cap\beta$ for some α and β.

2. In a triangle ABC, the perpendicular bisectors of sides AB and AC intersect BC at X and Y. Prove that $BC = XY$ if and only if $\tan B \tan C = 3$ or $\tan B \tan C = -1$.

3. Prove that for any natural number n and real numbers α and x satisfying $\alpha^{n+1} \le x \le 1$ and $0 < \alpha < 1$ it holds that
$$\prod_{k=1}^{n}\left|\frac{x-\alpha^k}{x+\alpha^k}\right| \le \prod_{k=1}^{n}\frac{1-\alpha^k}{1+\alpha^k}.$$

Second Day

4. On a table are given n markers, each of which is denoted by an integer. At any time, if some two markers are denoted with the same number, say k, we can redenote one of them with $k+1$ and the other one with $k-1$. Prove that after a finite number of moves all the markers will be denoted with different numbers.

5. Determine all pairs of integers (x,y) satisfying the equation
$$x^3+x^2y+xy^2+y^3 = 8(x^2+xy+y^2+1).$$

6. In a tetrahedron of volume V the sum of the squares of the lengths of its edges equals S. Prove that
$$V \le \frac{S\sqrt{S}}{72\sqrt{3}}.$$

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com