25-th Polish Mathematical Olympiad 1973/74

Third Round

First Day

1. In a tetrahedron $ABCD$ the edges AB and CD are perpendicular and $\angle ACB = \angle ADB$. Prove that the plane through AB and the midpoint of the edge CD, is perpendicular to CD.

2. A salmon in a mountain river must overpass two waterfalls. In every minute, the probability of the salmon to overpass the first waterfall is $p > 0$, and the probability to overpass the second waterfall is $q > 0$. These two events are assumed to be independent. Compute the probability that the salmon did not overpass the first waterfall in n minutes, assuming that it did not overpass both waterfalls in that time.

3. Let r be a natural number. Prove that the quadratic trinomial $x^2 - rx - 1$ does not divide any nonzero polynomial whose coefficients are integers with absolute values less than r.

Second Day

4. Prove that for every natural number n and a sequence of real numbers $a_1, a_2, \dots, a_n$ there exists a natural number k that satisfies
$$\left|\sum_{i=1}^{k} a_i - \sum_{i=k+1}^{n} a_i\right| \le \max_{1 \le i \le n} |a_i|.$$

5. Prove that for any natural numbers n, r with $r + 3 \le n$ the binomial coefficients $\binom{n}{r}, \binom{n}{r+1}, \binom{n}{r+2}, \binom{n}{r+3}$ cannot be successive terms of an arithmetic progression.

6. Several diagonals in a convex n-gon are drawn so as to divide the n-gon into triangles and:

   (i) the number of diagonals drawn at each vertex is even;

   (ii) no two of the diagonals have a common interior point.

   Prove that n is divisible by 3.

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