57-th Polish Mathematical Olympiad 2005/06

Third Round

Piotrków Trybunalski, April 5–6, 2006

First Day

1. solve in real numbers a, b, c, d, e the system of equations:

$$a^2 = b^3 + c^3,$$
$$b^2 = c^3 + d^3,$$
$$c^2 = d^3 + e^3,$$
$$d^2 = e^3 + a^3,$$
$$e^2 = a^3 + b^3.$$

2. Find all positive integers k for which the number $3^k + 5^k$ is a power of an integer with the exponent greater than 1.

3. A convex hexagon $ABCDE$ with $AC = DF$, $CE = FB$, and $EA = BD$ is given. Prove that the lines joining the midpoints of opposite sides of this hexagon meet in a point.

Second Day

4. The following operation is performed on a triple of numbers. Two of the numbers are chosen and replaced by their sum and product, while the third number is left unchanged. Decide whether, starting from the triple $(3, 4, 5)$ and performing finitely many such operations, we can obtain another triple of numbers which are the side lengths of a right triangle.

5. The inscribed sphere of a tetrahedron $ABCD$ with $AB = CD$ touches the faces ABC and ABD at K and L, respectively. Prove that if K and L are the centroids of the corresponding faces, then $ABCD$ is a regular tetrahedron.

6. Find all pairs of integers (a, b) for which there exists a polynomial $P(x)$ with integer coefficients such that the product $(x^2 + ax + b)P(x)$ is a polynomial of the form
$$x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0,$$
where each of $c_0, \ldots, c_{n-1}$ is equal to 1 or -1.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com