56-th Polish Mathematical Olympiad 2004/05

Third Round

April 13–14, 2005

First Day

1. Find all triples (x,y,n) of positive integers satisfying the equation

$$(x-y)^n = xy.$$

2. A convex quadrilateral $ABCD$ is inscribed in a circle o. Point S inside the circle is such that $\angle SAD = \angle SCB$ and $\angle SDA = \angle SBC$. The bisector of angle ASB intersects the circle o at points P and Q. Prove that $PS = QS$.

3. In a $2n \times 2n$ board ($n \in \mathbb{N}$) are written $4n^2$ real numbers with the sum 0 (one number in each cell). The absolute value of any of the numbers does not exceed 1. Prove that the absolute value of all numbers in a column or a row does not exceed n.

Second Day

4. A real number $c > -2$ is given. Prove that if positive numbers $x_1, x_2, \dots, x_n$ satisfy

$$\sqrt{x_1^2 + cx_1x_2 + x_2^2} + \sqrt{x_2^2 + cx_2x_3 + x_3^2} + \cdots + \sqrt{x_n^2 + cx_nx_1 + x_1^2} = \sqrt{c+2}(x_1 + x_2 + \cdots + x_n),$$

then $c = 2$ or $x_1 = x_2 = \cdots = x_n$.

5. Let $k > 1$ be an integer, and let $m = 4k^2 - 5$. Show that there exist positive integers a and b such that the sequence (x_n) defined by

$$x_0 = a, \quad x_1 = b, \quad x_{n+2} = x_{n+1} + x_n \quad \text{for} \quad n = 0, 1, 2, \dots$$

has all of its terms relatively prime to m

6. Show that every convex hexagon of area 1 contains a convex hexagon of area not smaller than $3/4$.