55-th Polish Mathematical Olympiad 2003/04

Third Round

April 15–16, 2004

First Day

1. A point D is taken on the side AB of a triangle ABC. Two circles passing through D and touching AC and BC at A and B respectively intersect again at point E. Let F be the point symmetric to C with respect to the perpendicular bisector of AB. Prove that the points D, E, and F lie on a line.

2. Let W be a polynomial with integer coefficients such that there are two distinct integers at which W takes coprime values. Show that there exists an infinite set of integers, such that the values W takes at them are pairwise coprime.

3. On a tournament with $n \geq 3$ participants, every two participants played exactly one match and there were no draws. A three-element set of participants is called a *draw-triple* if they can be enumerated so that the first defeated the second, the second defeated the third, and the third defeated the first. Determine the largest possible number of draw-triples on such a tournament.

Second Day

4. If a, b, c are real numbers, prove that
$$\sqrt{2(a^2+b^2)}+\sqrt{2(b^2+c^2)}+\sqrt{2(c^2+a^2)} \geq \sqrt{3(a+b)^2+3(b+c)^2+3(c+a)^2}.$$

5. Find the greatest possible number of lines in space that all pass through a single point and the angle between any two of them is the same.

6. An integer $m > 1$ is given. The infinite sequence $x_0, x_1, x_2, \dots$ is defined by
$$x_i = \begin{cases} 2^i & \text{for } i < m, \\ x_{i-1} + x_{i-2} + \cdots + x_{i-m} & \text{for } i \geq m. \end{cases}$$
Find the largest natural number k for which there exist k successive terms of this sequence which are divisible by m.

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