50-th Polish Mathematical Olympiad 1998/99

Second Round

February 26–27, 1999

First Day

1. Let $f:(0,1)\to\mathbb{R}$ be a function such that $f(1/n)=(-1)^n$ for all $n\in\mathbb{N}$. Prove that there are no increasing functions $g,h:(0,1)\to\mathbb{R}$ such that $f=g-h$.

2. A cube of edge 2 with one of the corner unit cubes removed is called a *piece*. Prove that if a cube T of edge 2^n is divided into 2^{3n} unit cubes and one of the unit cubes is removed, then the rest can be cut into pieces.

3. Let $ABCD$ be a cyclic quadrilateral and let E and F be the points on the sides AB and CD respectively such that $AE:EB=CF:FD$. Point P on the segment EF satsfies $EP:PF=AB:CD$. Prove that the ratio of the areas of $\triangle APD$ and $\triangle BPC$ does not depend on the choice of E and F.

Second Day

4. Let P be a point inside a triangle ABC such that $\angle PAB=\angle PCA$ and $\angle PAC=\angle PBA$. If $O\neq P$ is the circumcenter of $\triangle ABC$, prove that $\angle APO$ is right.

5. Let $S=\{1,2,3,4,5\}$. Find the number of functions $f:S\to S$ such that $f^{50}(x)=x$ for all $x\in S$.

6. Suppose that $a_1,a_2,\dots,a_n$ are integers such that
$$a_1+2^ia_2+3^ia_3+\cdots+n^ia_n=0 \quad \text{for } i=1,2,\dots,k-1,$$
where $k\geq 2$ is a given integer. Prove that $a_1+2^ka_2+3^ka_3+\cdots+n^ka_n$ is divisible by $k!$.

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