45-th Polish Mathematical Olympiad 1993/94

Second Round

February 1994

First Day

1. Find all real polynomials $P(x)$ of degree 5 such that $(x-1)^3 \mid P(x)+1$ and $(x+1)^3 \mid P(x)-1$.

2. Let $a_1,\dots,a_n$ be positive real numbers such that $\sum_{i=1}^{n} a_i = \prod_{i=1}^{n} a_i$, and let $b_1,\dots,b_n$ be positive real numbers such that $a_i \le b_i$ for all i.

   Prove that $\sum_{i=1}^{n} b_i \le \prod_{i=1}^{n} b_i$

3. A plane passing through tht center of a cube intersects the cube in a cyclic hexagon. Show that this hexagon is regular.

Second Day

4. Each vertex of a cube is assigned 1 or -1. Each face is assigned the product of the four numbers at its vertices. Determine all possible values that can be obtained as the sum of all the 14 assigned numbers.

5. The incircle o of a triangle ABC is tangent to the sides AB and BC at P and Q respectively. The angle bisector at A meets PQ at point S. Prove $\angle ASC = 90^\circ$.

6. Let p be a prime number. Prove that there exists $n \in \mathbb{Z}$ such that $p \mid n^2-n+3$ if and only if there exists $m \in \mathbb{Z}$ such that $p \mid m^2-m+25$.