58-th Polish Mathematical Olympiad 2006/07

First Round

September 11 – December 9, 2006

1. Find all triples (x,y,z) of real numbers satisfying the equations

$$\begin{aligned} x^2+2yz+5x &= 2 \\ y^2+2zx+5y &= 2 \\ z^2+2xy+5z &= 2. \end{aligned}$$

2. Find all pairs (k,m) of positive integers for which k^2+4m and m^2+5k are both perfect squares.

3. Let $ABCD$ be a convex quadrilateral with $AB = CD$ which is not a parallelogram. Points M and N are the midpoints of diagonals AC and BD respectively. Prove that the orthogonal projections of segments AB and CD on the line MN both have length equal to MN.

4. For every integer $n \geq 3$ determine the number of sequences $(c_1, c_2, \dots, c_n)$ with terms in $S = \{0,1,2,\dots,9\}$ which satisfy the following condition: For every three consequent terms, at least two of them are equal.

5. In an acute triangle ABC the angle at C is equal to 45°, O is the circumcenter and H the orthocenter. The line through O perpendicular to CO meets the lines AC and BC at K and L, respectively. Prove that $OK+KH = OL+LH$.

6. Let a,b,c be positive numbers. Prove the inequality

$$\frac{1}{a+ab+abc}+\frac{1}{b+bc+bca}+\frac{1}{c+ca+cab} \leq \frac{1}{3\sqrt[3]{abc}}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

7. In a tetrahedron $ABCD$, Q is the intersection point of the bisector of $\angle ABC$ with AC, and P is the point symmetric to D with respect to Q. Point R on segment AB is such that $BR = \frac{1}{2}BC$. Prove that there exists a triangle with side lengths equal to BP, CD, and $2QR$.

8. Let p be a prime number. Show that there exists a permutation $(x_1, x_2, \dots, x_{p-1})$ of the numbers $1,2,\dots,p-1\}$ such that the numbers

$$x_1, x_1x_2, x_1x_2x_3,\ \dots,\ x_1x_2\cdots x_{p-1}$$

are distinct modulo p.

9. For a natural number k, denote by $F(k)$ the product of all divisors of k. Prove or disprove that there exist two different numbers $m,n \in \mathbb{N}$ with $F(m) = F(n)$.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com

10. Let ABC be an acute triangle. Points P and U are on segment BC, Q and S on segment CA, and R and T on segment AB, such that

$$PR \perp BC,\ \ QP \perp CA,\ \ RQ \perp AB,\ \ US \perp BC,\ \ ST \perp CA,\ \ TU \perp AB.$$

Prove that the triangles PQR and STU are congruent.

11. For each natural number n find the number of permutations $(x_1, x_2, \dots, x_{6n-1})$ of the set $\{1, 2, \dots, 6n-1\}$ with the following properties:

    (i) If $i - j = 2n + 1$, then $x_i > x_j$;

    (ii) If $i - j = 4n$, then $x_i < x_j$.

12. A polynomial W with real coefficients takes only positive values on the segment $[a, b]$ ($a < b$). Show that there exist polynomials P and $Q_1, Q_2, \dots, Q_m$ such that

$$W(x) = P(x)^2 + (x-a)(b-x)\sum_{i=1}^{m} Q_i(x)^2 \quad \text{for all real } x.$$