54-th Polish Mathematical Olympiad 2002/03

First Round

September – December, 2002

1. Find all pairs of positive integers x,y such that $(x+y)^2-2(xy)^2=1$.

2. Given a real number a_1, define the sequence (a_n) by $a_{n+1}=a_n^2-a_n+1$ for $n\geq 1$. Prove that for all positive integers n,
$$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}<\frac{1}{a_1-1}.$$

3. Three different points A,B,C are given on a circle o. The tangent lines to o at A and B meet at P, and the tangent line to o at C intersects the line AB at Q. Prove that $PQ^2=PB^2+QC^2$.

4. Consider the set of all sequences of length k with terms in the set $\{1,2,\dots,m\}$. For each of these sequences the value of the smallest term is marked. Prove that the sum of all the marked numbers is equal to $1^k+2^k+\cdots+m^k$.

5. A positive integer n_1 contains 333 decimal digits, and all these digits are nonzero. For $i=1,2,\dots,332$, set n_{i+1} to be the number obtained from n_i by moving the last digit of n_i to the beginning. Prove that 333 divides either none, or all of the numbers $n_1,n_2,\dots,n_{333}$.

6. Points A,B,C,D lie in this order on a circle o. Let M be the midpoint of the arc AB of o not containing C,D, and N be that of the arc CD not containing A,B. Prove that
$$\frac{AN^2-BN^2}{AB}=\frac{DM^2-CM^2}{CD}.$$

7. On a meeting at aunt Renia met n persons (including the aunt). Each person gave at least one gift to at least one other person. Each person except aunt Renia gave thrice as many gifts as he/she received, but aunt Renia received six times more gifts as she gave out. Find the smallest number of gifts which aunt Renia could have obtained.

8. In a tetrahedron $ABCD$, M and N are the midpoints of the edges AB and CD, respectively. Suppose that a point P on segment MN satisfies $MP=CN$ and $NP=AC$. Let O be the circumcenter of the tetrahedron. Show that if $O\neq P$, then $OP\perp MN$.

9. Find all polynomials W with real coefficients having the following property: If $x+y$ is a rational number, then so is $W(x)+W(y)$.

10. A deck of 52 cards labelled with numbers $1,2,\dots,52$ is given. A permutation π of the set $\{1,2,\dots,52\}$ is called a *shuffle* if there is an integer $1\leq m\leq 51$ such that $\pi(i)<\pi(i+1)$ for $i=1,2,\dots,m-1,m+1,m+2,\dots,51$. Prove or disprove

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com

that starting with cards in an arbitrary order we can arrange them in any other order in at most 5 shuffles.

11. A convex quadrilateral $ABCD$ is given. Points P and Q different from its vertices lie on the segments BC and CD, respectively, and satisfy the condition $\angle BAP = \angle DAQ$. Show that the triangles ABP and ADQ have the same area if and only if their orthocenters lie on a line perpendicular to AC.

12. For positive real numbers a, b, c, d, denote $A = a^3 + b^3 + c^3 + d^3$ and $B = bcd + cda + dab + abc$. Prove that

$$(a+b+c+d)^3 \leq 4A + 24B.$$