52-nd Polish Mathematical Olympiad 2000/01

First Round

September–December, 2000

1. Solve in integers the equation $x^{2000} + 2000^{1999} = x^{1999} + 2000^{2000}$.

2. Points D and E lie on the sides BC and AC respectively of a triangle ABC. The lines AD and BE meet at P. Points K and L are taken on BC and AC respectively so that $CLPK$ is a parallelogram. Prove that $\dfrac{AE}{EL} = \dfrac{BD}{DK}$.

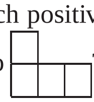
3. Find all integers $n \geq 2$ such that the inequality
$$x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n \leq \frac{n-1}{n}(x_1^2 + \cdots + x_n^2)$$
is satisfied for all positive numbers $x_1, x_2, \ldots, x_n$.

4. Prove or disprove: One can place 65 balls of diameter 1 within a cube box of edge 4.

5. Prove that for all integers $n \geq 2$ and all prime numbers p the number $n^{p^p} + p^p$ is composite.

6. The integers a, b, x, y satisfy the equality
$$a + b\sqrt{2001} = \left(x + y\sqrt{2001}\right)^{2000}.$$
Prove that $a \geq 44b$.

7. Points D and E lie on the hypotenuse BC of an isosceles right triangle ABC such that $\angle DAE = 45^\circ$. The circumcircle of triangle ADE meets the sides AB and AC again at P and Q, respectively. Prove that $BP + CQ = PQ$.

8. For which positive integers m, n can an $m \times n$ rectangle be cut into pieces congruent to ?

9. Prove that among any 12 consecutive integers there is one that cannot be written as a sum of 10 fourth powers.

10. Prove that each triangle ABC contains an interior point P with the following property: each line passing through P divides the perimeter and the area of $\triangle ABC$ in the same ratio.

11. An n-tuple $(c_1, c_2, \ldots, c_n)$ of positive integers is called *admissible* if each positive integer k not exceeding $2(c_1 + c_2 + \cdots + c_n)\}$ can be represented in the form
$$k = \sum_{i=1}^{n} a_i c_i, \quad \text{with } a_i \in \{-2, -1, 0, 1, 2\}.$$
For each n find the maximum possible value of $c_1 + \cdots + c_n$ if $(c_1, \ldots, c_n)$ is admissible.

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12. Consider all sequences $x_0, x_1, \ldots, x_{2000}$ of integers satisfying

$$x_0 = 0 \quad \text{and} \quad |x_n| = |x_{n-1} + 1| \text{ for } n = 1, 2, \ldots, 2000.$$

Find the minimum value of the expression $|x_1 + x_2 + \cdots + x_{2000}|$.

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