Dutch Mathematical Olympiad 1993

Second Round

September 17

1. Show that any subset of $V = \{1,2,\dots,24,25\}$ with 17 or more elements contains at least two distinct numbers the product of which is a perfect square.

2. In a triangle ABC with $\angle A = 90^\circ$, D is the midpoint of BC, F that of AB, E that of AF and G that of FB. Segment AD intersects CE, CF and CG in P, Q an R, respectively. Determine the ratio PQ/QR.

3. A sequence of numbers is defined by $u_1 = a$, $u_2 = b$ and $u_{n+1} = \dfrac{u_n + u_{n-1}}{2}$ for $n \geq 2$. Prove that $\lim\limits_{n\to\infty} u_n$ exists and express its value in terms of a and b.

4. Let C be a circle with center M in a plane V, and P be a point not on the circle C.

   (a) If P is fixed, prove that $AP^2 + BP^2$ is a constant for every diameter AB of the circle C.

   (b) Let AB be a fixed diameter of C and P a point on a fixed sphere S not intersecting V. Determine the points P on S that minimize $AP^2 + BP^2$.

5. Eleven distinct points $P_1, P_2, \dots, P_{11}$ are given on a line so that $P_iP_j \leq 1$ for every i, j. Prove that the sum of all distances P_iP_j, $1 \leq i < j \leq 11$, is smaller than 30.

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