Dutch Mathematical Olympiad 1990

Second Round

September 9

1. Prove that for every integer $n > 1$, $1 \cdot 3 \cdot 5 \cdots (2n-1) < n^n$.

2. Consider the sequence $a_1 = \dfrac{3}{2}$, $a_{n+1} = \dfrac{3a_n^2 + 4a_n - 3}{4a_n^2}$.

   (a) Prove that $1 < a_n$ and $a_{n+1} < a_n$ for all n.

   (b) From (a) it follows that $\lim_{n\to\infty} a_n$ exists. Find this limit.

   (c) Determine $\lim_{n\to\infty} a_1 a_2 a_3 \cdots a_n$.

3. A polynomial $f(x) = ax^4 + bx^3 + cx^2 + dx$ with $a,b,c,d > 0$ is such that $f(x)$ is an integer for $x \in \{-2,-1,0,1,2\}$ and $f(1) = 1$ and $f(5) = 70$.

   (a) Show that $a = \frac{1}{24}$, $b = \frac{1}{4}$, $c = \frac{11}{24}$, $d = \frac{1}{4}$.

   (b) Prove that $f(x)$ is an integer for all $x \in \mathbb{Z}$.

4. If $ABCDEFG$ is a regular 7-gon with side 1, show that $\dfrac{1}{AC} + \dfrac{1}{AD} = 1$.

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