7-th Nordic Mathematical Contest

March 17, 1993

1. Let F be an increasing real function defined for $0 \le x \le 1$ such that

   (i) $F(x/3) = F(x)/2$ and

   (ii) $F(1-x) = 1 - F(x)$ for all x.

   Find $F\left(\frac{173}{1993}\right)$ and $F\left(\frac{1}{13}\right)$.

2. A hexagon is inscribed in a circle with radius r. Two sides of hexagon have length 1, two have length 2, and two have length 3. Prove that r is a root of the equation
$$2r^3 - 7r - 3 = 0.$$

3. Let $s(t)$ denote the number of digits of a natural number t. Find all solutions to the system
$$\begin{aligned} s(x) + s(y) &= x, \\ x + y + s(z) &= z, \\ s(x) + s(y) + s(z) &= y - 4. \end{aligned}$$

4. Let $T(n)$ be the sum of the decimal digits of a natural number n.

   (a) Find a natural number N such that $T(kN)$ is even for all $k = 1, 2, \ldots, 1992$, but not for $k = 1993$.

   (b) Show that there is no N such that $T(kN)$ is even for all $k \in \mathbb{N}$.

1