36-th Mongolian Mathematical Olympiad 2000

Final Round

Sukhbaatar, May 1–6

Grade 10

First Day

1. Let $\operatorname{rad}(k)$ denote the product of prime divisors of a natural number k (define $\operatorname{rad}(1)=1$). A sequence (a_n) is defined by setting a_1 arbitrarily, and $a_{n+1}=a_n+\operatorname{rad}(a_n)$ for $n\ge 1$. Prove that the sequence (a_n) contains arithmetic progressions of arbitrary length.

2. Circles $\omega_1,\omega_2,\omega_3$ with centers O_1,O_2,O_3, respectively, are externally tangent to each other. The circle ω_1 touches ω_2 at P_1 and ω_3 at P_2. For any point A on ω_1, A_1 denotes the point symmetric to A with respect to O_1. Show that the intersection points of AP_2 with ω_3, A_1P_3 with ω_2, and AP_3 with A_1P_2 lie on a line.

3. A cube of side n is cut into n^3 unit cubes, and m of these cubes are marked so that the centers of any three marked cubes do not form a right-angled triangle with legs parallel to sides of the cube. Find the maximum possible value of m.

Second Day

4. Suppose that a function $f:\mathbb{R}\to\mathbb{R}$ satisfies the following conditions:

   (i) $|f(a)-f(b)|\le|a-b|$ for all $a,b\in\mathbb{R}$;

   (ii) $f(f(f(0)))=0$.

   Prove that $f(0)=0$.

5. Given a natural number n, find the number of quadruples (x,y,u,v) of integers with $1\le x,y,u,v\le n$ satisfying the following inequalities:

$$\begin{aligned}1&\le v+x-y\le n,\\1&\le x+y-u\le n,\\1&\le u+v-y\le n,\\1&\le v+x-u\le n.\end{aligned}$$

6. In a triangle ABC, the angle bisector at A,B,C meet the opposite sides at A_1,B_1,C_1, respectively. Prove that if the quadrilateral $BA_1B_1C_1$ is cyclic, then

$$\frac{AC}{AB+BC}=\frac{AB}{AC+CB}+\frac{BC}{BA+AC}.$$

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Teachers – secondary level

First Day

1. Find all integers that can be written in the form $\dfrac{(x+y+z)^2}{xyz}$, where x,y,z are positive integers.

2. Let $n \geq 2$. For any two n-vectors $\vec{x} = (x_1, \dots, x_n)$ and $\vec{y} = (y_1, \dots, y_n)$, we define
$$f(\vec{x},\vec{y}) = x_1\overline{y_1} - \sum_{i=2}^{n} x_i\overline{y_i}.$$
Prove that if $f(\vec{x},\vec{x}) \geq 0$ and $f(\vec{y},\vec{y}) \geq 0$, then $|f(\vec{x},\vec{y})|^2 \geq f(\vec{x},\vec{x})f(\vec{y},\vec{y})$.

3. Two points A and B move around two different circles in the plane with the same angular velocity. Suppose that there is a point C which is equidistant from A and B at every moment. Prove that, at some moment, A and B will coincide.

Second Day

4. In a country with n towns, the distance between the towns numbered i and j is denoted by x_{ij}. Suppose that the total length of every cyclic route which passes through every town exactly once is the same. Prove that there exist numbers a_i, b_i $(i = 1, \dots, n)$ such that $x_{ij} = a_i + b_j$ for all distinct i, j.

5. Let m, n, k be positive integers with $m \geq 2$ and $k \geq \log_2(m-1)$. Prove that
$$\prod_{s=1}^{n} \frac{ms-1}{ms} < \sqrt[2^{k+1}]{\frac{1}{2n+1}}.$$

6. Given distinct prime numbers $p_1, \dots, p_s$ and a positive integer n, find the number of positive integers not exceeding n that are divisible by exactly one of the p_i.

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