12-th Mexican Mathematical Olympiad

Querétaro, Querétaro, November 1998

First Day

1. A number is called *lucky* if computing the sum of the squares of its digits and repeating this operation sufficiently many times leads to number 1. For example, 1900 is lucky, as $1900 \mapsto 82 \mapsto 68 \mapsto 100 \mapsto 1$. Find infinitely many pairs of consecutive numbers each of which is lucky.

2. Rays l and m forming an angle of α are drawn from the same point. Let P be a fixed point on l. For each circle $\mathscr{C}$ tangent to l at P and intersecting m at Q and R, let T be the intersection point of the bisector of angle QPR with $\mathscr{C}$. Describe the locus of T and justify your answer.

3. Each side and each diagonal of a regular octagon is colored red or black. Show that there are at least seven triangles formed by vertices of the octagon and having all sides of the same color.

Second Day

4. Find all integers that can be written in the form
$$\frac{1}{a_1} + \frac{2}{a_2} + \cdots + \frac{9}{a_9},$$
where $a_1, a_2, \dots, a_9$ are nonzero digits, not necessarily different.

5. The tangents at points B and C on a given circle meet at point A. Let Q be a point on segment AC and let BQ meet the circle again at P. The line through Q parallel to AB intersects BC at J. Prove that PJ is parallel to AC if and only if $BC^2 = AC \cdot QC$.

6. A plane in space is *equidistant* from a set of points if its distances from the points in the set are equal. What is the largest possible number of equidistant planes from five points, no four of which are coplanar?

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