10-th Mexican Mathematical Olympiad 1996.

Mérida, Yucatán – November 1996

First Day

1. Let P and Q be the points on the diagonal BD of a quadrilateral $ABCD$ such that $BP = PQ = QD$. Let AP and BC meet at E, and let AQ meet DC at F.
   (a) Prove that if $ABCD$ is a parallelogram, then E and F are the midpoints of the corresponding sides.
   (b) Prove the converse of (a).

2. There are 64 booths around a circular table and on each one there is a chip. The chips and the corresponding booths are numbered 1 to 64 in this order. At the center of the table there are 1996 light bulbs which are all turned off. Every minute the chips move simultaneously in a circular way (following the numbering sense) as follows: chip 1 moves one booth, chip 2 moves two booths, etc., so that more than one chip can be in the same booth. At any minute, for each chip sharing a booth with chip 1 a bulb is lit. Where is chip 1 on the first minute in which all bulbs are lit?

3. Prove that it is not possible to cover a 6×6 square board with eighteen 2×1 rectangles, in such a way that each of the lines going along the interior gridlines cuts at least one of the rectangles. Show also that it is possible to cover a 6×5 rectangle with fifteen 2×1 rectangles so that the above condition is fulfilled.

Second Day

4. For which integers $n \geq 2$ can the numbers 1 to 16 be written each in one square of a squared 4×4 paper such that the 8 sums of the numbers in rows and columns are all different and divisible by n?

5. The numbers 1 to n^2 are written in an $n \times n$ squared paper in the usual ordering. Any sequence of steps from a square to an adjacent one (by side) starting at square 1 and ending at square n^2 is called a *path*. Denote by $\mathscr{L}(\mathscr{C})$ the sum of the numbers through which path $\mathscr{C}$ goes.
   (a) For a fixed n, let M and m be the largest and smallest $\mathscr{L}(\mathscr{C})$ possible. Prove that $M - m$ is a perfect square.
   (b) Prove that for no n can one find a path $\mathscr{C}$ with $\mathscr{L}(\mathscr{C}) = 1996$.

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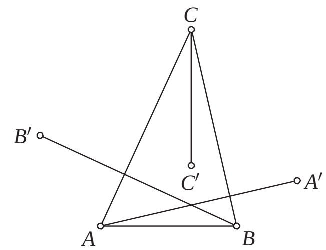
6. In a triangle ABC with $AB < BC < AC$, points A', B', C' are such that $AA' \perp BC$ and $AA' = BC$, $BB' \perp CA$ and $BB' = CA$, and $CC' \perp AB$ and $CC' = AB$, as shown on the picture. Suppose that $\angle AC'B$ is a right angle. Prove that the points A', B', C' are collinear.



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