5-th Mexican Mathematical Olympiad

Oaxtepec, Morelos, November 1991

First Day

1. Evaluate the sum of all positive irreducible fractions less than 1 and having the denominator 1991.

2. A company of n soldiers is such that

   (i) n is a palindrome number (read equally in both directions);

   (ii) if the soldiers arrange in rows of 3, 4 or 5 soldiers, then the last row contains 2, 3 and 5 soldiers, respectively.

   Find the smallest n satisfying these conditions and prove that there are infinitely many such numbers n.

3. Four balls of radius 1 are placed in space so that each of them touches the other three. What is the radius of the smallest sphere containing all of them?

Second Day

1. The diagonals AC and BD of a convex quarilateral $ABCD$ are orthogonal. Let M,N,R,S be the midpoints of the sides AB,BC,CD and DA respectively, and let W,X,Y,Z be the projections of the points M,N,R and S on the lines CD,DA,AB and BC, respectively. Prove that the points M,N,R,S,W,X,Y and Z lie on a circle.

2. The sum of squares of two consecutive integers can be a square, as in $3^2+4^2=5^2$. Prove that the sum of squares of m consecutive integers cannot be a square for $m=3$ or 6 and find an example of 11 consecutive integers the sum of whose squares is a square.

3. Given an n-gon ($n \geq 4$), consider a set $\mathscr{T}$ of triangles formed by vertices of the polygon having the following property: Every two triangles in $\mathscr{T}$ have either two common vertices, or none. Prove that $\mathscr{T}$ contains at most n triangles.

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