6-th Mediterranean Mathematical Competition 2003

1. Prove that the equation $x^2 + y^2 + z^2 = x + y + z + 1$ has no rational solutions.

2. In a triangle ABC with $BC = CA + \frac{1}{2}AB$, point P is given on side AB such that $BP : PA = 1 : 3$. Prove that $\angle CAP = 2\angle CPA$.

3. Let a, b, c be nonnegative numbers with $a + b + c = 3$. Prove the inequality

$$\frac{a}{b^2 + 1} + \frac{b}{c^2 + 1} + \frac{c}{a^2 + 1} \geq \frac{3}{2}.$$

4. Consider a system of infinitely many spheres made of metal, with centers at points $(a, b, c) \in \mathbb{R}^3$. We say that the system is *stable* if the temperature of each sphere equals the average temperature of the six closest spheres. Assuming that all spheres in a stable system have temperatures between $0°$C and $1°$C, prove that all the spheres have the same temperature.