Moldovan Team Selection Tests 2009

First Test

1. Find the smallest $n \in \mathbb{N}$ for which there exists $m \in \mathbb{N}$, such that the rectangle $(3m+2)\times(4m+3)$ can be covered with $n(n+1)/2$ squares, such that n of the squares are of the side 1, $n-1$ of side 2, ..., 1 square of length n. Find a covering for such n.

2. Given two integers m, n such that $m \geq 1$, $n \geq 2$, assume that a_i $(i = 1,\dots,n)$ are real numbers with the sum 1. Prove that

$$\frac{a_1^{2-m}+a_2+\cdots+a_{n-1}}{1-a_1}+\frac{a_2^{2-m}+a_3+\cdots+a_n}{1-a_2}+ \\ +\cdots+\frac{a_n^{2-m}+a_1+\cdots+a_{n-2}}{1-a_n} \quad \geq \quad n+\frac{n^m-n}{n-1}.$$

3. Assume that the diagonal BD is a diameter of the circle circumscribed about $ABCD$. Let A_1 be the reflection of A with respect to BD, and B_1 the reflection of B with respect to AC. Denote by P the intersection of CA_1 with BD, and by Q the intersection of DB_1 with AC. Prove that $AC \perp PQ$.

4. Let p be a prime divisor of positive integer n which satisfy $n \geq 2$. Prove that there exists a set $A = \{a_1,\dots,a_n\} \subseteq \mathbb{N}$ such that the product of any two elements of A is divisible by the sum of any p numbers from A.

Second Test

1. Given a positive integer n, solve the equation

$$\left\{\left(x+\frac{1}{m}\right)^3\right\} = x^3.$$

2. Determine all functions $f : [0,+\infty) \to [0,+\infty$ such that:

$$f(x+y-z)+f(2\sqrt{xz})+f(2\sqrt{yz}) = f(x+y+z)$$

for all real $x,y,z \geq 0$ such that $x+y \geq z$.

3. Given two circles Ω_1 and Ω_2, assume that the radius of Ω_2 is larger than the radius of Ω_1 and that the circles tangent each other from the outside. A line t_1 touches Ω_1 at A and Ω_2 at D. Line t_2 parallel to t_1 is tangent to Ω_1 and intersects Ω_2 at E and F. Let C be the point on Ω_2 that is on the other side of EF than D. Let B be the intersection of EF and CD. Prove that the circumcircle of $\triangle ABC$ is tangent to the line AD.

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4. Assume that $m, n \in \mathbb{N}$, and that in every cell of the $2m \times 2n$ table there is either $+$ or $-$. Define a *cross* to be the union of all cells from one row and one column of the table. A cell that is in the intersection of this row and column is called the *center* of the cross. Consider the following transformation: First we mark all points with the sign $-$. Then, for every marked cell we change the signs in the cross whose center is the chosen cell. We call the table *accessible* if it can be obtained from another table after one transformation. Find the number of accessible tables.

Third Test

1. Points X, Y, and Z are located on the sides BC, CA, and AB of $\triangle ABC$ such that $\triangle XYZ \sim \triangle ABC$. Prove that the circumcircle of $\triangle AYZ$ passes through a fixed point.

2. Let M be a set of arithmetic progressions with integer terms and ratio bigger than 1.

   (a) Prove that the set $\mathbb{Z}$ can be expressed as a union of the finite number of the progressions from M with different ratios.

   (b) Prove that $\mathbb{Z}$ can not be written as a union of the finite number of the progressions from M with relatively prime integer ratios.

3. A weightlifter uses a weight that has two sides each of which consists of n small weights. At each stage she takes some weights from one of the sides in such a way that at any moment the difference of the numbers of weights on the sides does not exceed k. What is the minimal number of stages (as a function of n and k) necessary to remove all the weights?

4. Let x, y, and z be real numbers from the interval $[1/2, 2]$. Let (a, b, c) be a permutation of $\{x, y, z\}$. Prove that:

$$\frac{60a^2-1}{4xy+5z} + \frac{60b^2-1}{4yz+5x} + \frac{60c^2-1}{4zx+5y} \geq 12.$$

Fourth Test

1. Let $ABCD$ be a trapezoid with $AB \| CD$. Let E and F be the points in its exterior such that $\triangle ABE$ and $\triangle CDF$ are equilateral. Prove that the lines AC, BD, and EF are concurrent.

2. Let f and g be two polynomials with nonzero degree and integer coefficients, such that $g \mid f$ and $f + 2009$ has 50 integer roots. Prove that the degree of g is at least 5.

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3. The sequence $(a_n)_{n\in\mathbb{N}}$ is defined as follows:

$$a_n = \frac{2}{3+1} + \frac{2^2}{3^2+1} + \frac{2^3}{3^4+1} + \cdots + \frac{2^{n+1}}{3^{2^n}+1}.$$

Prove that $a_n < 1$ for any $n \in \mathbb{N}$.

4. In a group of people each two are either friends or enemies. Each pair of friends doesn't have common friends, and each pair of enemies has exactly two common friends. Prove that each person from the group has equal number of friends.

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