42-nd Moldova Mathematical Olympiad 1998

Final Round – Chişinău, March 16–17

Grades 7 and 8

First Day

1. Eight keys are provided for eight rooms in which an olympiad will be organized. However, it is not indicated which key corresponds to which room. What is the minimum number of trials that guarantees opening all the rooms?

2. Let D and E be the midpoints of the sides AB and BC of a triangle ABC. Point M is taken on side AC so that $ME > CE$. Prove that $AD > MD$.

3. Păcală and Tândală received their annual salaries in banknotes of 13 lei (the currency of Moldova). In a restaurant, Păcală orders 9 pieces of bread, 10 glasses of juice, and 7 sausages, and Tândală orders 5 pieces of bread, 7 glasses of juice, and one sausage. The prices of a piece of bread, juice and sausage are integer numbers of lei. Show that if Păcală can pay his order with an integer number of banknotes without change, then Tândală can do so as well.

4. Two casks with capacities of a and b liters were filled up with juices of different prices per liter. We simultaneously take 999 liters of juice from each cask and pour it into the other cask. The resulting mixtures in the two casks have equal price per liter. If a and b are integers with the smallest possible sum, find a and b.

Second Day

5. There are n pupils and a box with candies. The first pupil took one candy and one tenth of the rest. Then the second pupil took two candies and one tenth of the rest, and so on. It turned out that all pupils took the same number of candies. How many pupils and candies were there?

6. A car moves with the speed 90 km/h down the vales, 72 km/h on plain and 60 km/h uphill. The car covers the distance from town A to town B in 5 hours and from town B to town A in 4 hours. Find the distance between the towns A and B.

7. Let $A = \overline{a_1 \dots a_{n-1} a_n}$ and $B = \overline{a_1 \dots a_{n-1}} + 4a_n$ be the decimal representations of two natural numbers A, B. Prove that A is divisible by 13 if and only if so is B.

8. In a convex pentagon $ABCDE$, K, L, M, N are the midpoints of sides AB, AE, CD, BC, and P and Q are the midpoints of KM and LN, respectively. Prove that PQ is parallel to DE and $PQ = \frac{1}{4}DE$.

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Grade 9

First Day

1. For all real numbers a,b,c,d,e, prove the inequality

$$a^2+b^2+c^2+d^2+e^2 \geq ab+ac+ad+ae.$$

2. Let n be a positive integer and $A_n = 2^{2n}(2^{2n+1}-1)$.
   (a) Find the units decimal digit of A_n.
   (b) Show that the tens digit of A_n is 2.

3. For each triangle ABC with a fixed perimeter $2p$, the tangent to the incircle of the triangle parallel to BC and distinct from BC meets the sides of the triangle at M and N.
   (a) Prove that the length of MN is maximal when $AC+AB=3BC$.
   (b) Show that in this case the ratio AI/IA_1 does not depend on the triangle, where I is the incenter and A_1 the intersection point of AI and BC.

4. Given a pair (x,y) of real numbers, each of the pairs $(x+1,y+2x+1)$, $(\frac{x}{y},\frac{1}{y})$ for $y\neq 0$ and $(\alpha x,\alpha^2 y)$ for $\alpha\in\mathbb{R}\setminus\{0\}$ can be obtained. Starting from the pair $(35,1998)$, can one obtain the pair $(3,4)$?

Second Day

5. At a school party which the whole class attended, whenever a boy dances with some girl for the first time, he gives her a candy. This way the first girl received 4 candies, the second one 5 candies, the third one 6, and so on. All the girls received 130 candies in total. It is known that the last girl danced with each boy. How many girls and boys were there in the class?

6. We are given 1998 weights with the masses $1,2,\dots,1998$. For which n from the set $\{2,3,4,5\}$ can the weights be divided into n groups with the same mass?

7. A pupil chooses an integer, multiplies it with 0.42 and rounds it up to an integer, and then again multiplies the result by 0.42 and rounds it up to an integer, obtaining 8. Which number did he choose?

8. Circles k_1 and k_2 intersect each other, and a circle k_3 with center O touches k_1 and k_2 internally at A and B respectively. A line is tangent to k_1 at C and to k_2 at D. Let E be the foot of the perpendicular from O to CD. Show that the lines AC,BD, and OE are concurrent.

Grade 10

First Day

1. Solve in real numbers the equation $\log_2(1+\sqrt{x}) = \log_3 x$.

2. Solve in real numbers the equation $\left[\frac{2x-1}{3x+2}\right] = \frac{80}{9} + \frac{4}{3}x - 4x^2$.

3. In a triangle ABC the bisector of angle A meets the side BC at A_1 and the circumcircle again at A_2. Points B_1, B_2 and C_1, C_2 are analogously defined. Show that
$$\frac{A_1A_2}{BA_2 + A_2C} + \frac{B_1B_2}{CB_2 + B_2A} + \frac{C_1C_2}{AC_2 + C_2B} \geq \frac{3}{4}.$$

4. A cube with side 45 is divided into unit cube cells, 1998 of which are populated by bacteria. Later on the bacteria occupy each cell which shares a face with at least three occupied cells. Can the bacteria occupy all of the large cube?

Second Day

5. Prove that for any positive integer n the number
$$E = 666^n + 648^n + (-1)^{n+1}684^n$$
is divisible by 1998.

6. A sequence $(a_n)_{n=1}^{\infty}$ satisfies $a_1 + a_2 + \cdots + a_n = n^2 a_n$ for each n. For which values of a_1 is a_{1998} an integer?

7. A set A of integers has the property that for all $x, y \in A$, $x - y$ also belongs to A. Suppose that $1998 \in A$ and that the segment $[-100, 100]$ contains not less than 33 and not more than 66 elements of A. How many elements of A does the segment $[-1998, 1998]$ contain?

8. A circle k is externally tangent to circles k_1 and k_2 with respective centers O_1 and O_2 at points A and B, respectively. A line touches k_1 at C and k_2 at D so that CD and AB are on different sides of line O_1O_2. Prove that the points A, B, C and D are concyclic.

Grade 11

First Day

1. The sequence (a_n) is defined by $a_1 = 19$, $a_2 = 98$ and
$$a_{n+2} = a_n - \frac{2}{a_{n+1}} \quad \text{for } n \geq 1.$$
Show that there exists a natural number m for which $a_m = 0$ and determine it.

2. Let $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ for $n \in \mathbb{N}$. Prove that for any $n > 2$,

$$a_n^2 > 2\left(\frac{a_2}{2} + \frac{a_3}{3} + \cdots + \frac{a_n}{n}\right).$$

3. Two parallel planes α and β are given. A sphere σ is tangent to α at point T. Two distinct lines a and b through T meet β at C and D and the sphere σ at B and A, respectively. Prove that the points A, B, C, D lie on a circle.

4. We are given a chessboard 1998×1998, colored in the usual way.

   (a) Show that one can choose several squares and put a coin on each so that every row or column contains exactly 42 coins.

   (b) If the coins are arranged in such a way, prove that the number of coins on black squares is even.

Second Day

5. A sequence is defined by

$$a_1 = \frac{1}{2} \quad \text{and} \quad a_n = \frac{a_{n-1}}{2na_{n-1}+1} \text{ for } n > 1.$$

   Evaluate $a_1 + a_2 + \cdots + a_{1998}$.

6. Find all functions $f : \mathbb{Q} \to \mathbb{R}$ with the property that $f(-8) = 0$ and

$$f(x+y) = f(x) + f(y) + 3xy(x+y+6) - 8 \quad \text{for all } x, y \in \mathbb{Q}.$$

7. Spheres s_1 and s_2 intersect each other, and a sphere s_3 with center O touches s_1 and s_2 internally at A and B respectively. A plane is tangent to s_1 at C and to s_2 at D. Let E be the foot of the perpendicular from O to this plane. Show that the lines AC, BD, and OE are concurrent.

8. A polynomial $P(x)$ with integer coefficients has an integral root. Prove that $P(1995)P(2000)$ cannot be a power of 7.

Grade 12

First Day

1. *Problem 1 for Grade 11.*

2. Find all polynomials $P(x)$ with real coefficients satisfying

$$P(x)^2 + P\left(\frac{1}{x}\right)^2 = P(x^2)P\left(\frac{1}{x^2}\right) \quad \text{for all } x > 0.$$

3. *Problem 3 for Grade 11.*

4. *Problem 4 for Grade 11.*

Second Day

5. *Problem 5 for Grade 11.*

6. *Problem 6 for Grade 11.*

7. *Problem 7 for Grade 11.*

8. (a) Can the set $X = [0,1]$ be partitioned into two nonempty subsets A and B for which there is a continuous function $f : X \to X$ such that $f(A) \subset B$ and $f(B) \subset A$?

   (b) The same question for $X = [0,1)$.

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