44-th Moldova Mathematical Olympiad 2000

Final Round – Chişinău, March 13–14

Grade 7

First Day

1. What is the greatest possible number of Fridays by the date 13 in a year?

2. Prove that if real numbers a,b,c,d satisfy $a^2+b^2+(a+b)^2=c^2+d^2+(c+d)^2$, then they also satisfy $a^4+b^4+(a+b)^4=c^4+d^4+(c+d)^4$.

3. Consider the sets $A_1=\{1\}$, $A_2=\{2,3,4\}$, $A_3=\{5,6,7,8,9\}$, etc. Let b_n be the arithmetic mean of the smallest and the greatest element in A_n. Show that the number $\frac{2000}{b_1-1}+\frac{2000}{b_2-1}+\cdots+\frac{2000}{b_{2000}-1}$ is a prime integer.

4. The orthocenter H of a triangle ABC is not on the sides of the triangle and the distance AH equals the circumradius of the triangle. Find the measure of $\angle A$.

Second Day

5. Several crocodiles, dragons and snakes were left on an island. Animals were eating each other according to the following rules. Every day at the breakfast, each snake ate one dragon; at the lunch, each dragon ate one crocodile; and at the dinner, each crocodile ate one snake. On the Saturday after the dinner, only one crocodile and no snakes and dragons remained on the island. How many crocodiles, dragons and snakes were there on the Monday in the same week before the breakfast?

6. A natural number $n\geq 5$ leaves the remainder 2 when divided by 3. Prove that the square of n is not a sum of a prime number and a perfect square.

7. The Fibonacci sequence is defined by $F_0=F_1=1$ and $F_{n+2}=F_{n+1}+F_n$ for $n\geq 0$. Prove that the sum of 2000 consecutive terms of the Fibonacci sequence is never a term of the sequence.

8. Points D and N on the sides AB and BC and points E,M on the side AC of an equilateral triangle ABC, respectively, with E between A and M, satisfy $AD+AE=CN+CM=BD+BN+EM$. Determine the angle between the lines DM and EN.

Grade 8

First Day

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D. Djukić, V. Janković, I. Matić, N. Petrović
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1. Find all positive integers a for which $a^{2000}-1$ is divisible by 10.

2. Thirty numbers are arranged on a circle in such a way that each number equals the absolute difference of its two neighbors. Given that the sum of the numbers is 2000, determine the numbers.

3. Suppose that $m,n \geq 2$ are integers such that $m+n-1$ divides m^2+n^2-1. Prove that the number $m+n-1$ is not prime.

4. Let $ABCDEF$ be a regular hexagon and P be a point on the shorter arc EF of its circumcircle. Prove that the value of

$$\frac{AP+BP+CP+DP}{EP+FP}$$

is constant and find this value.

Second Day

5. An airline offer 2000 two-way routes connecting 64 towns in a country. Show that it is possible to reach any town from any other town using the offered routes.

6. Assuming that real numbers x and y satisfy $y(1+x^2)=x\left(\sqrt{1-4y^2}-1\right)$, find the maximum value of xy.

7. For any real number a, prove the following inequality:

$$(a^3+a^2+3)^2 > 4a^3(a-1)^2.$$

8. A rectangular parallelepiped has dimensions a,b,c that satisfy the relation $3a+4b+10c=500$, and the length of the main diagonal $20\sqrt{5}$. Find the volume and the total area of the surface of the parallelepiped.

Grade 9

First Day

1. Let a,b,c be real numbers with $a,c \neq 0$. Prove that if r is a real root of $ax^2+bx+c=0$ and s a real root of $-ax^2+bx+c=0$, then there is a root of $\frac{a}{2}x^2+bx+c=0$ between r and s.

2. Prove that if a,b,c are integers with $a+b+c=0$, then $2a^4+2b^4+2c^4$ is a perfect square.

3. The diagonals of a convex quadrilateral $ABCD$ are orthogonal and intersect at a point E. Prove that the projections of E on AB,BC,CD,DA are concyclic.

4. A rectangular field consists of 1520 unit squares. How many rectangles 6×1 at most can be cut out from this field?

Second Day

5. Solve in real numbers the equation

$$(x^2-3x-2)^2-3(x^2-3x-2)-2-x=0.$$

6. Find all nonnegative integers n for which n^8-n^2 is not divisible by 72.

7. In a trapezoid $ABCD$ with $AB \parallel CD$, the diagonals AC and BD meet at O. Let M and N be the centers of the regular hexagons constructed on the sides AB and CD in the exterior of the trapezoid. Prove that M, O and N are collinear.

8. Initially the number 2000 is written down. The following operation is repeatedly performed: the sum of the 10-th powers of the last number is written down. Prove that in the infinite sequence thus obtained, some two numbers will be equal.

Grade 10

First Day

1. Suppose that real numbers x, y, z satisfy

$$\frac{\cos x+\cos y+\cos z}{\cos(x+y+z)}=\frac{\sin x+\sin y+\sin z}{\sin(x+y+z)}=p.$$

Prove that $\cos(x+y)+\cos(y+z)+\cos(x+z)=p$.

2. Show that if real numbers $x<1<y$ satisfy the inequality

$$2\lg x+\lg(1-x)\geq 3\lg y+\lg(y-1),$$

then $x^3+y^3<2$.

3. For every nonempty subset X of $M=\{1,2,\ldots,2000\}$, a_X denotes the sum of the minimum and maximum element of X. Compute the arithmetic mean of the numbers a_X when X goes over all nonempty subsets X of M.

4. Let $A_1A_2\ldots A_n$ be a regular hexagon and M be a point on the shorter arc A_1A_n of its circumcircle. Prove that the value of

$$\frac{A_2M+A_3M+\cdots+A_{n-1}M}{A_1M+A_nM}$$

is constant and find this value.

Second Day

5. Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy $f(x+y) - f(x-y) = 2y(3x^2+y^2)$ for all $x, y \in \mathbb{R}$.

6. Find all real values of the parameter a for which the system

$$\begin{aligned} 1+(4x^2-12x+9)^2+2^{y+2} &= a \\ \log_3\left(x^2-3x+\frac{117}{4}\right)+32 &= a+\log_3(2y+3) \end{aligned}$$

has a unique real solution. Solve the system for these values of a.

7. In an isosceles triangle ABC with $BC = AC$, I is the incenter and O the circumcenter. The line through I parallel to AC meets BC at D. Prove that the lines DO and BI are perpendicular.

8. Two circles intersect at M and N. A line through M meets the circles at A and B, with M between A and B. Let C and D be the midpoints of the arcs AN and BN not containing M, respectively, and K and L be the midpoints of AB and CD, respectively. Prove that $CL = KL$.

Grade 11

First Day

1. Positive numnbers a and b satisfy $a^{1999}+b^{2000} \geq a^{2000}+b^{2001}$. Prove that $a^{2000}+b^{2000} \leq 2$.

2. Solve the system

$$\begin{aligned} 36x^2y-27y^3 &= 8, \\ 4x^3-27xy^2 &= 4. \end{aligned}$$

3. The excircle of a triangle ABC corresponding to A touches the side BC at M, and the point on the incircle diametrically opposite to its point of tangency with BC is denoted by N. Prove that A, M, and N are collinear.

4. Find all polynomials $P(x)$ with real coefficients that satisfy the relation

$$1+P(x) = \frac{P(x-1)+P(x+1)}{2}.$$

Second Day

5. For a positive integer p, define the function $f : \mathbb{N} \to \mathbb{N}$ by $f(n) = a_1^p + \cdots + a_m^p$, where $a_1, \ldots, a_m$ are the decimal digits of n. Prove that every sequence $(b_k)_{k=0}^{\infty}$ of positive integers, which satisfies $b_{k+1} = f(b_k)$ for all k, has a finite number of distinct terms.

6. Let $(a_n)_{n\geq 0}$ be a sequence of positive numbers that satisfy the relations $a_{i-1}a_{i+1} \leq a_i^2$ for all $i \in \mathbb{N}$. For any integer $n > 1$, prove the inequality
$$\frac{a_0+\cdots+a_n}{n+1}\cdot\frac{a_1+\cdots+a_{n-1}}{n-1} \geq \frac{a_0+\cdots+a_{n-1}}{n}\cdot\frac{a_1+\cdots+a_n}{n}.$$

7. A triangle whose all sides have lengths greater than 1 is contained in a unit square. Show that the center of the square lies inside the triangle.

8. In an isosceles triangle ABC with $BC = AC$ and $\angle B < 60^\circ$, I is the incenter and O the circumcenter. The circle with center E that passes through A, O and I intersects the circumcircle of $\triangle ABC$ again at point D. Prove that the lines DE and BO intersect on the circumcircle of ABC.

Grade 12

First Day

1. Let $1 = d_1 < d_2 < \cdots < d_{2m} = n$ be the divisors of a positive integer n, where n is not a perfect square. Consider the determinant
$$D = \begin{vmatrix} n+d_1 & n & \cdots & n \\ n & n+d_2 & \cdots & n \\ \cdots & \cdots & & \cdots \\ n & n & \cdots & n+d_{2m} \end{vmatrix}.$$
   (a) Prove that n^m divides D;
   (b) Prove that $1+d_1+d_2+\cdots+d_{2m}$ divides D.

2. For $n \in \mathbb{N}$, define
$$a_n = \frac{1}{\binom{n}{1}} + \frac{1}{\binom{n}{2}} + \cdots + \frac{1}{\binom{n}{n}}.$$
   (a) Prove that the sequence $b_n = a_n^n$ is convergent and compute its limit.
   (b) Show that $\lim_{n\to\infty} b_n > \left(\frac{3}{2}\right)^{\sqrt{3}+\sqrt{2}}$.

3. For any $n \in \mathbb{N}$, denote by a_n the sum $2+22+222+\cdots+22\ldots2$, where the last summand consists of n digits 2. Determine the greatest n for which a_n contains exactly 222 digits 2.

4. Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that $\int_0^1 x^m f(x) = 0$ for $m = 0, 1, \ldots, 1999$. Prove that f has at least 2000 zeroes on the segment $[0,1]$.

Second Day

5. Prove that there is no polynomial $P(x)$ with real coefficients that satisfies

$$P'(x)P''(x) > P(x)P'''(x) \quad \text{for all } x \in \mathbb{R}.$$

Is this statement true for the thrice differentiable real functions?

6. Show that there is a positive number p such that $\int_0^\pi x^p \sin x \, dx = \sqrt[10]{2000}$.

7. Prove that for any positive integer n there exists a matrix of the form

$$A = \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & a & b \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

(a) with nonzero entries,

(b) with positive entries,

such that all the entries of A^n are perfect squares.

8. A circle with radius r touches the sides AB, BC, CD, DA of a convex quadrilateral $ABCD$ at E, F, G, H, respectively. The inradii of the triangles EBF, FCG, GDH, HAE are equal to r_1, r_2, r_3, r_4. Prove that

$$r_1 + r_2 + r_3 + r_4 \geq 2(2 - \sqrt{2})r.$$

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