5-th Macedonian Mathematical Olympiad 1998

Skopje, April 11, 1998

1. Let $ABCDE$ be a convex pentagon with $AB = BC = CA$ and $CD = DE = EC$. Let T be the centroid of $\triangle ABC$, and N be the midpoint of AE. Compute $\angle NTD$.

2. Prove that the numbers $1, 2, \ldots, 1998$ cannot be separated into three classes whose sums of elements are divisible by $2000, 3999$, and 5998, respectively.

3. A triangle ABC is given. For every positive numbers p, q, r, let A', B', C' be the points such that $\overrightarrow{BA'} = p\overrightarrow{AB}$, $\overrightarrow{CB'} = q\overrightarrow{BC}$, and $\overrightarrow{AC'} = r\overrightarrow{CA}$. Define $f(p,q,r)$ as the ratio of the area of $\triangle A'B'C'$ to that of $\triangle ABC$. Prove that for all positive numbers x, y, z and every positive integer n,

$$\sum_{k=0}^{n-1} f(x+k, y+k, z+k) = n^3 f\left(\frac{x}{n}, \frac{y}{n}, \frac{z}{n}\right).$$

4. If P is the area of a triangle ABC with sides a, b, c, prove that

$$\frac{ab+bc+ca}{4P} \geq \sqrt{3}.$$

5. The sequence (a_n) is defined by $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 - \sqrt{4 - a_n^2}}$. Let $b_n = 2^{n+1} a_n$. Prove that $b_n \leq 7$ and $b_n < b_{n+1}$ for all n.

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