15-th Macedonian Mathematical Olympiad 2008

April 19

1. Find all injective functions $f:\mathbb{N}\to\mathbb{N}$ which satisfy
$$f(f(n)) \le \frac{n+f(n)}{2} \quad \text{for each } n\in\mathbb{N}.$$

2. Positive numbers a,b,c are such that $(a+b)(b+c)(c+a)=8$. Prove the inequality
$$\frac{a+b+c}{3} \ge \sqrt[27]{\frac{a^3+b^3+c^3}{3}}.$$

3. An acute triangle ABC with $AB\ne AC$ is given. Let V and D be the feet of the altitude and angle bisector from A, and let E and F be the intersection points of the circumcircle of $\triangle AVD$ with sides AC and AB respectively. Prove that AD, BE and CF have a common point.

4. We call an integer $n>1$ *good* if, for any natural numbers $1\le b_1,b_2,\dots,b_{n-1}\le n-1$ and any $i\in\{0,1,\dots,n-1\}$, there is a subset I of $\{1,\dots,n-1\}$ such that $\sum_{k\in I} b_k \equiv i \pmod{n}$. (The sum over the empty set is zero.) Find all good numbers.

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