8-th Macedonian Mathematical Olympiad 2001

1. Prove that if m and s are integers with $ms = 2000^{2001}$, then the equation $mx^2 - sy^2 = 3$ has no integer solutions.

2. Does there exist a function $f : \mathbb{N} \to \mathbb{N}$ such that
$$f(f(n-1)) = f(n+1) - f(n) \quad \text{for all } n \geq 2?$$

3. Let ABC be a scalene triangle and k be its circumcircle. Let t_A, t_B, t_C be the tangents to k at A, B, C, respectively. Prove that points $AB \cap t_C$, $AC \cap t_B$, and $BC \cap t_A$ exist, and that they are collinear.

4. Let Ω be a family of subsets of M such that:

   (i) If $|A \cap B| \geq 2$ for $A, B \in \Omega$, then $A = B$;

   (ii) There exist different subsets $A, B, C \in \Omega$ with $|A \cap B \cap C| = 1$;

   (iii) For every $A \in \Omega$ and $a \in M \setminus A$, there is a unique $B \in \Omega$ such that $a \in B$ and $A \cap B = \emptyset$.

   Prove that there are numbers p and s such that:

   (1) Each $a \in M$ is contained in exactly p sets in Ω;

   (2) $|A| = s$ for all $A \in \Omega$;

   (3) $s + 1 \geq p$.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com