12-th Korean Mathematical Olympiad 1998/99

Final Round

First Day – April 17, 1999

1. Let R and r be the circumradius and inradius of $\triangle ABC$, and R' and r' be those of $\triangle A'B'C'$. Prove that if $\angle C = \angle C'$ and $Rr' = R'r$, then the two triangles are similar.

2. Suppose $f(x)$ is a function satisfying $|f(m+n) - f(m)| \leq \frac{n}{m}$ for all natural numbers m, n. Show that for all natural numbers k,

$$\sum_{i=1}^{k} |f(2^k) - f(2^i)| \leq \frac{k(k-1)}{2}.$$

3. Find all positive integers n such that $2^n - 1$ is a multiple of 3 and $\dfrac{2^n - 1}{3}$ divides $4m^2 + 1$ for some integer m.

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4. Suppose that a function $f(x)$ satisfies

$$f\left(\frac{x-3}{x+1}\right) + f\left(\frac{3+x}{1-x}\right) = x \quad \text{for all } x \text{ with } |x| \neq 1.$$

   Find all such functions f.

5. A permutation $a_1, a_2, \ldots, a_6$ of numbers $1, 2, \ldots, 6$ can be transformed to $1, 2, \ldots, 6$ by transposing two numbers exactly four times. Find the number of such permutations.

6. Let $a_1, a_2, \ldots, a_{1999}$ be nonnegative real numbers satisfying:

   (i) $a_1 + a_2 + \cdots + a_{1999} = 2$;
   (ii) $a_1a_2 + a_2a_3 + \cdots + a_{1998}a_{1999} + a_{1999}a_1 = 1$.

   Find the maximum and minimum values of $S = a_1^2 + a_2^2 + \cdots + a_{1999}^2$.

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